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ELEMENTS

OF

ALGEBRA.

DESIGNED FOR

SCHOOLS, ACADEMIES, AND COLLEGES.

BY

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PREFACE.

Perhaps every teacher of experience has formed in his own mind a conception of what he regarded as an unexceptionable text book on some favorite branch of study. The author of this treatise has had several years' experience as a teacher of mathematics, and during this time, he has given several of the more important works on algebra a fair trial in the recitation-room, the only proper place to test the merits of a school book. This trial has resulted in the belief, that a work on algebra, adapted to the wants of the majority of those who make this science a study, is yet a desideratum. For this reason, the following treatise has been prepared.

Great care has been taken to adopt such demonstrations of theorems, and discussions of principles, as are at once characterized by the greatest simplicity, rigor, and brevity. No effort has been made to simplify a demonstration, or discussion, by entering into mere matters of detail; for, it is believed, that such attempts to adapt everything to the comprehension of the meanest capacity, often fail in their object, and cause many an intelligent pupil to lose sight of important principles, as he is groping his way through a multiplicity of words.

This work is, as its title indicates, an Elementary Treatise on Algebra; but, it must not therefore be inferred, that it may be studied without requiring any intellectual exertion. It has not been written for the purpose of pleasing "mental dyspeptics whose minds loathe vigorous thought as a diseased stomach loathes wholesome food." Such a work might suit the taste of some, but its puerilities would displease every intelligent scholar. Throughout the whole work, it is hoped that the student will find opportunities for exercising his powers of thought and analysis.

Our experience in teaching algebra has proved that the scholar can only acquire a familiarity with principles by applying these principles to the solution of practical questions. For this reason, a very large number of practical examples and problems will be found

in this work. Many of these examples are not contained in any other school book. In selecting these examples, it has been made a prominent object to select such as would be most likely to interest the student, and at the same time, to have them of such a nature, that their solutions would strengthen and increase his powers of analysis.

On the Theory of Equations we have been concise, but it is hoped that that part of it of which we have treated will be found amply sufficient for most practical purposes. We will refer the reader to the table of contents for a brief enumeration of the topics discussed in this work.

The utility of the study of Algebra cannot be questioned. Indeed, the study of any branch of mathematics is admirably calculated to give tone and vigor to the mental operations. Since the invention of representing by equations, lines, surfaces, and solids in geometry, the study of algebra has acquired a very great importance. It may be said to constitute the basis of sound mathematical learning. Should this volume be the means of facilitating the progress of the student in an important science, the wishes of the Author will be gratified.

CORTLAND ACADEMY,
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CHAPTER I.

DEFINITIONS AND NOTATION.

(**1.**) ALGEBRA is that method of computation in which letters and other symbols are employed.

(**2.**) The first letters of the alphabet are generally employed to denote known quantities, and the last, those which are unknown.

(**3.**) The symbol $=$ denotes equality; as, $a=b$, which is read, a equals b.

(**4.**) The symbol $+$ denotes addition; as, $a+b$, or a added to b. Those quantities to which this sign is prefixed, are called *positive quantities.*

(**5.**) The symbol $-$ denotes subtraction; as, $a-b$, or b subtracted from a. Those quantities to which this sign is prefixed, are called *negative quantities.*

(**6.**) The symbol $\times$ denotes multiplication; as, $a\times b$, which is read, a multiplied by b.

(**7.**) The symbol $\div$ denotes division; as, $a\div b$, which is read, a divided by b.

(**8.**) The symbol $>$ denotes inequality; as, $a>b$, which is read a is greater than b. The greater quantity is always placed next to the opening, as in the above example.

(9.) The use of the parenthesis, in Algebra, may be illustrated by an example. If we wish to express, *algebraically*, the product of the sum of two quantities, a and b, by the difference of two quantities, x and y, we may do it thus, $(a+b)\times(x-y)$. A straight line placed over a quantity is called a *vinculum*. It is used for the same purpose as the parenthesis. Thus, $\overline{a+b}\times\overline{x-y}$ is the same as the former expression.

(10.) A *co-efficient* is a number or letter which is used to denote how many times a quantity is taken. Thus, in the expression $3x$, 3 is the co-efficient of x. In the expression ax, a is the co-efficient of x, and it shows that x is to be taken as many times as there are units in a.

(11.) An *exponent* is a number or letter which is used to denote how many times a quantity must be taken as a factor. Thus, the expression a^3 is the same as $a\times a\times a$. When an exponent is *fractional*, the denominator of the exponent shows the number of *equal factors* into which the quantity must be *divided;* and the numerator shows how many of these equal factors must be taken. Thus, in the expression $8^{\frac{2}{3}}$, the denominator 3, shows that 8 is to be resolved into *three equal factors*, and the numerator 2, shows that two of these equal factors must be taken. Now 2 is one of the equal factors of 8; hence, $8^{\frac{2}{3}}=2\times2=4$.

(12.) One of the two equal factors of a quantity is called the *square root* of that quantity. One of three equal factors of a quantity is called the *cube root* of that quantity. In general, the nth root of a quantity is one of the n equal factors into which that quantity may be resolved.

(13.) Roots are frequently denoted by fractional exponents; as $a^{\frac{1}{3}}$, which is read the cube root of a. $8^{\frac{2}{3}}$ is read the cube root of the square of 8, or 64. The symbol $\sqrt{\ }$ denotes the square root. Thus, $a^{\frac{1}{2}}$ and $\sqrt{a}$ are the same.

(14.) When no exponent is expressed, the exponent 1 is always understood. The exponent or index of a quantity is written a little above and to the right of the letter; as, a^4.

(15.) When a quantity is multiplied by itself, the product is called the *square*, or second power of that quantity. When a quantity is used as a factor three times, the product is called the *cube* of that quantity. And, in general, when a quantity is used as a factor n times, the product thus obtained is called the nth power of that quantity.

(16.) The *reciprocal* of a quantity is a unit divided by that quantity; thus, $\frac{1}{a}$ is the reciprocal of a.

(17.) The symbol $\therefore$ signifies *therefore* or *consequently.*

(18.) An algebraical expression consisting of two or more terms connected by the signs $+$ or $-$, is called a *polynomial.* An algebraical expression having only one term is called a *monomial.* When a polynomial consists of two terms it is called a *binomial;* one consisting of three terms is called a *trinomial.* Thus, $a+b$ is a binomial, and $a+b-c$ is a trinomial; $a+b-3c+4d+h$ is a polynomial.

(19.) Every quantity written in algebraical characters is called an *algebraical expression.* Thus, $3a^2$ is an algebraical expression for three times the square of the number a.

(20.) When each term of a polynomial contains the same number of literal factors, the polynomial is said to be *homogeneous.* a^2+ax+y^2 is a homogeneous polynomial of the second degree, because *each* term contains *two* factors.

(21.) The product of several letters is indicated by writing the letters one after the other. Thus, the expression ab is the same as $a\times b$. A period is sometimes used as a sign of multiplication; as $a.b$, which is read, a multiplied by b.

(22.) *Similar terms* are those which contain the same letters, affected with the same exponents. *Dissimilar terms* are those which do not contain the same letters, affected with the same exponents.

(23.) An *axiom* is a self-evident truth.

(24.) A *theorem* is a truth which is made evident by means of a process of reasoning called a demonstration. A theorem is often called a proposition.

(25.) A *problem* is a question which requires something to be ascertained.

(26.) A *corollary* is an obvious consequence drawn from some proposition.

(27.) A *lemma* is a proposition which is employed to establish some proposition which immediately follows it.

CHAPTER II.

ADDITION.

(28.) Addition is the finding of an algebraical expression for the sum of several quantities.

There are three cases in the addition of algebraical quantities.

CASE I.

(29.) *When the quantities are similar, and have the same sign.*

If we wish to add $5ac^2+4b$ to $7ac^2+6b$, we can say that $5ac^2$ and $7ac^2$ are $12ac^2$, and $4b$ and $6b$ are $10b$, $\therefore$ the sum is $12ac^2+10b$.

Operation.

$$\begin{array}{r} 5ac^2+\ \ 4b \\ 7ac^2+\ \ 6b \\ \hline 12ac^2+10b \end{array} \text{ sum.}$$

Whence we have the following

RULE.

Arrange the similar terms under each other, and then add the co-efficients of similar terms, and annex the common literal quantity.

EXAMPLES.

1. What is the sum of $8b+7c$, $4b+19c$, and $17b$?

Ans. $29b+26c$.

2. What is the sum of $4ax^2+19c^2$, $23ax^2+4c^2$, and $17c^2$?

Ans. $27ax^2+40c^2$.

3. What is the sum of $16xy+17ay^2$, $23xy+27ay^2$, and $47ay^2$?

Ans. $39xy+91ay^2$.

4. What is the sum of $14z^2+10ab$, $26z^2+17ab$, and $13z^2+5ab$? *Ans.*

CASE II.

(30.) *When the quantities are similar, and have different signs.*

If we wish to add $10ab+14c^2$ to $6ab-8c^2$, we can say that $10ab$ and $6ab$ are $16ab$. Now, the sign $-$ before $8c^2$ indicates that it must be *subtracted* and not added. $8c^2$ taken from $14c^2$ leaves $+6c^2$. Hence the sum is $16ab+6c^2$.

Operation.

$$\begin{array}{r} 10ab+14c^2 \\ 6ab-\ \ 8c^2 \\ \hline 16ab+\ \ 6c^2 \end{array}$$

Hence, when the quantities are similar, and have different signs, we have this

RULE.

Arrange the similar terms under each other, and then find the sum of the co-efficients of the positive quantities, and also the sum of the co-efficients of the negative quantities, in each set of similar terms; to the difference of these sums affix the common literal quantity, and prefix the sign of the greater sum.

EXAMPLES.

1. What is the sum of $x^3-3x^2y+3xy^2-y^3$ and $x^3+3x^2y+3xy^2+y^3$? *Ans.* $2x^3+6xy^2$.

2. What is the sum of $cx^2+10cx^2-18ab+4cx^2+10ab-4cx^2+4ab$? *Ans.* $11cx^2-4ab$.

3. What is the sum of $5a^{\frac{2}{3}}+6ay-15a^{\frac{2}{3}}+4ay-3a^{\frac{2}{3}}$?

Ans. $-13a^{\frac{2}{3}}+10ay$.*

4. What is the sum of $my^{\frac{1}{2}}+6x^3+10my^3-4x^3-15my^3$?

Ans. $my^{\frac{1}{2}}+2x^3-5my^3$.

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* NOTE.—$5a^{\frac{2}{3}}$ is read five times the cube root of the square of a. See Art. 18.

5. What is the sum of $16xy-4hm+17xy+43hm$?
Ans. $33xy+39hm$.

6. What is the sum of $5c^4-49an+10c^4+14an$?
Ans. $15c^4-35an$.

7. What is the sum of $10x^{\frac{1}{2}}y^{\frac{1}{4}}-17sk+15x^{\frac{1}{2}}y^{\frac{1}{4}}+5sk-4x^{\frac{1}{2}}y^{\frac{1}{4}}$?
Ans. $21x^{\frac{1}{2}}y^{\frac{1}{4}}-12sk$.

8. What is the sum of $20a^2c^2x+15ah-15a^2c^2x-23ah$?
Ans. $5a^2c^2x-8ah$.

9. What is the sum of $4x^2-3x+4$, $x-2x^2-5$, $1+3x^2-5x$, $2x-4+7x^2$, $13-x^2-4x$?
Ans. $11x^2-9x+9$.

10. What is the sum of $4x^3-2x+y$, $4x-y-x^3$, $9y+7x^3-x$, and $21x-2y+9x^3$?
Ans. $19x^3+22x+7y$.

11. What is the sum of $4-3x$, $x-5$, $2x-6$, $-4x+13$, and $-5x+1$?
Ans. $7-9x$.

12. What is the sum of $a^3-b^3+3a^2b-5ab^2$, $3a^3-4a^2b+3b^3-3ab^2$, $a^3+b^3+3a^2b$, $2a^3-4b^3-5ab^2$, $6a^2b+10ab^2$, and $-6a^3-7a^2b+4ab^2+2b^3$?
Ans. $a^3+a^2b+ab^2+b^3$.

13. What is the sum of $11bc+4ad-8ac+5cd$, $8ac+7bc-2ad+4mn$, $2cd-3ab+5ac+an$, and $9an-2bc-2ad+5cd$?
Ans. $16bc+5ac+12cd+4mn-3ab+10an$.

14. What is the sum of $x^4-4x^3y+6x^2y^2-4xy^3+y^4$, and $x^4+4x^3y+6x^2y^2+4xy^3+y^4$?
Ans. $2x^4+12x^2y^2+2y^4$.

15. What is the sum of $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$, and $y^5-5y^4x+10y^3x^2-10y^2x^3+5yx^4-x^5$?
Ans. $2y^5+20y^3x^2+10yx^4$.*

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* NOTE.—The pupil must observe that the value of the expression $5x^4y$ is the same as the value of $5yx^4$. The factors y and x^4, are only differently arranged. Therefore the sum of these two expressions is $10x^4y$ or $10yx^4$.

16. What is the sum of $3(a+b)^{\frac{1}{2}}+4a^2x^2-10$, and $5\sqrt{a+b}$ $+11a^2x^2+6$?

Ans. $8(a+b)^{\frac{1}{2}}+15a^2x^2-4$.* Or, $8\sqrt{a+b}+15a^2x^2-4$.

17. What is the sum of $a^2x^2-2acx+c^2$ and $a^2x^2+2acx+c^2$?

Ans. $2a^2x^2+2c^2$.

18. What is the sum of $x^3-ax^2+a^2x-a^3$ and $x^3+3ax^2+3a^2x$ $+a^3$?

Ans. $2x^3+2ax^2+4a^2x$.

(31.) In the preceding examples, the co-efficients have been *numeral.* When the co-efficients are *literal,* find the sum of the co-efficients which have a *common* literal quantity, enclose this sum in a parenthesis, and prefix it to the common literal quantity.

19. What is the sum of $ax^2+16bx^2+7cy^2-4ny^2$? In this example the common literal quantities are x^2 and y^2. The sum of the co-efficients of x^2 is $(a+16b)$, and the sum of the co-efficients of y^2 is $(7c-4n)$. Hence, the answer to this example is $(a+16b)x^2+(7c-4n)y^2$. If, in this example, $a=4$, $b=1$, $c=4$, and $n=6$, the last expression becomes, by making the substitution, $(4+16)x^2+(28-24)y^2=20x^2-4y^2$. In solving examples of this kind, it is convenient to arrange those terms which have a common literal quantity under each other.

Operation.

$$\begin{array}{l} ax^2+7cy^2 \\ 16bx^2-4ny^2 \\ \hline (a+16b)x^2+(7c-4n)y^2 \text{ ans.} \end{array}$$

20. What is the sum of $5axy+10abz$, and $15cz-2cxy$?

Ans. $(5a-2c)xy+(10ab+15c)z$.

21. What is the sum of $ax^{\frac{2}{3}}+bx^{\frac{2}{3}}$, and mx^4+nx^4?

Ans. $(a+b)x^{\frac{2}{3}}+(m+n)x^4$.

22. What is the sum of ax^2+by+k, and $dx^2+hy+ck$?

Ans. $(a+d)x^2+(b+h)y+(c+1)k$.

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* NOTE.—See Art. 13.

23. What is the sum of $5ca^2x^2+4ba^2x^2+mx^2y^2$, and $10ca^2x^2-2ba^2x^2+6mx^2y^2$?

Ans. $15ca^2x^2+2ba^2x^2+7mx^2y^2$; or, it may be written, $(15c+2b)a^2x^2+7mx^2y^2$.

24. What is the sum of $a^2x^2+b^2x^2$, and $9a^2x^2-11b^2x^2$?

Ans. $(10a^2-10b^2)x^2$.

If, in this example, we make $a=3$, $b=2$, and, consequently $a^2=9$, $b^2=4$, the expression $a^2x^2+b^2x^2$ becomes $9x^2+4x^2$, and the expression $9a^2x^2-11b^2x^2$ becomes $81x^2-44x^2$. Hence, $(a^2x^2+b^2x^2)+(9a^2x^2-11b^2x^2)=(9x^2+4x^2)+(81x^2-44x^2)=13x^2+37x^2=50x^2$. The co-efficient 50, may be obtained by substituting the values of a and b in the co-efficient of x^2 in the answer. $10a^2=10\times3^2=90$. $10b^2=10\times2^2=40$. Hence, $10a^2-10b^2=90-40=50$. This is only verifying the answer.

CASE III.

(32.) *When the quantities to be added are all dissimilar, or when some are similar and some dissimilar.*

If we wish to add a to b we can only indicate the addition by connecting the two quantities by the sign of addition. Thus, a added to b is expressed, $a+b$. And in any case, where all the quantities to be added are dissimilar, we can only add them by writing them, one after the other, with their respective signs. Thus, $ax+cy$ and $4nz$ are dissimilar quantities, and their sum is $ax+cy+4nz$.

If we wish to add $4a^2x^2+12cy$, and $6cy+10z+3a^2x^2$, we can, obviously, add the similar quantities, and write the dissimilar quantity $10z$ after their sum.

Operation.

$$\begin{array}{l} 4a^2x^2+12cy \\ \underline{3a^2x^2+\ \ 6cy+10z} \\ 7a^2x^2+18cy+10z \text{ sum.} \end{array}$$

For convenience in adding we have arranged the similar terms under each other. Hence, to add quantities all or part of which are dissimilar, we have the following

RULE.

Add the similar terms according to the directions given in Case I. and Case II., and then write the dissimilar terms one after the other, with their respective signs.

EXAMPLES.

1. What is the sum of $a^2x^2-6ab+y$, and $4a^2x^2+10ab+z$?
Ans. $5a^2x^2+4ab+y+z$.

2. What is the sum of $a^2+2ab+b^2$, and $a^2+2ax+x^2$?
Ans. $2a^2+2ab+b^2+2ax+x^2$.

3. What is the sum of $4a^2-4ab+4b^2$, and $a^2+2ab+b^2+y$?
Ans. $5a^2-2ab+5b^2+y$.

4. What is the sum of x^2+9x+9, and $-3x+10z$?
Ans. $x^2+6x+9+10z$.

5. What is the sum of $10m^2+16ab+10xy$, and $15ac^2b+7z$ $-12ab+4m^2$? *Ans.* $14m^2+4ab+10xy+15ac^2b+7z$.

SUBTRACTION.

(33.) SUBTRACTION, in Algebra, is the finding of the difference of two algebraic quantities.

If we subtract b from a the remainder is obviously $a-b$. Now, if we subtract $(b-c)$ from a, it is plain that the remainder ought to be greater than the first remainder by c, since the minuend is the *same*, and the subtrahend is *less* by c. Therefore the remainder is, $a-b+c$.

As another example, let it be required to subtract $20-5$ from 50. If we subtract 20, the remainder will obviously be too small by 5. Therefore we must add 5 to the remainder, $50-20$, in order to obtain the true remainder. Therefore the true remainder is, $50-20+5=35$.

Now, in each of these examples, we have simply changed the signs of the subtrahend, and added it, with the signs changed, to the minuend. Hence, for the subtraction of algebraic quantities, we have the following

RULE.

Change the signs of all the terms of the quantity to be subtracted, and then proceed as in addition.

EXAMPLES.

1. From $8x^2-3ax+5$ take $5x^2+2ax+5$.

Operation.

The minuend	$8x^2-3ax+5$
The subtrahend, with signs changed,	$-5x^2-2ax-5$
	$3x^2-5ax$ ans.

2. From $7xy-10y+4x$ take $3xy+3y+3x$.
Ans. $4xy-13y+x$.

3. From $a+b+c$ take $-a-b-c$. *Ans.* $2a+2b+2c$.

4. From $a^2x+6bx-9ax^4-7ay$ take $11ay-4a^2x+3bx-2ax^4$.
Ans. $5a^2x+3bx-7ax^4-18ay$.

5. From $17cx+12px-7ny^2$ take $14cx-7px+3ny^2-b$.
Ans. $3cx+19px-10ny^2+b$.

6. From $13xy-14xy^{\frac{1}{2}}+17ay$ take $4xy^{\frac{1}{2}}+7xy-10ay$.
Ans. $6xy-18xy^{\frac{1}{2}}+27ay$.

7. From $9a^2x^2-16+10ay^5$ take $4a^2x^2-8+7ay^5+y$.
Ans. $5a^2x^2-8+3ay^5-y$.

8. From $24xy^2-14my+18x^2y^2-14+27xz^2$ take $17xy^2-10my-4x^2y^2+20xz^2-8$. *Ans.* $7xy^2-4my+22x^2y^2+7xz^2-6$.

9. From $17pmx^2-18n^3+19m^4-24$ take $7pmx^2-4n^3+10m^4-17$. *Ans.* $10pmx^2-14n^3+9m^4-7$.

10. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$. *Ans.* $4ab$.

11. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$. *Ans.* $6a^2b+2b^3$.

12. From $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ take $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$. *Ans.* $4x^{\frac{1}{3}}y^{\frac{1}{3}}$.

13. From $6\sqrt{x+y}+4a^2x^2$ take $3(x+y)^{\frac{1}{2}}-4a^2x^2$. *Ans.* $3\sqrt{x+y}+8a^2x^2$.

14. From ax^2+ay^2 take bx^2+cy^2. *Ans.* $(a-b)x^2+(a-c)y^2$.*

15. From $ax^2-2cy+nk$ take $bx^2-4my+pk$. *Ans.* $(a-b)x^2+(4m-2c)y+(n-p)k$.

16. From $a^2x^2-4axy+4a^2y^2$ take $c^2x^2-8cxy+16x^2y^2$. *Ans.* $(a^2-c^2)x^2+(8c-4a)xy+(4a^2-16x^2)y^2$.

17. From $2x-y+(y-2x)-(x-2y)$ take $y-2x-(2y-x)+x+2y$. *Ans.* $y-x$.

18. From $ax^2+bxy+cy^2$ take $dx^2-hxy+hy^2$. *Ans.* $(a-d)x^2+(b+h)xy+(c-h)y^2$.

(34.) If we wish to indicate the subtraction of one polynomial from another, without actually performing the operation, we must enclose the polynomial to be subtracted within the parenthesis. Thus, $a^3+3a^2b+3ab^2+b^3-(a^3-3a^2b+3ab^2-b^3)$ signifies that the quantity within the parenthesis is to be subtracted from $a^3+3a^2b+3ab^2+b^3$. If the subtraction be actually performed by the rule, the remainder will be $6a^2b+2b^3$.

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* NOTE.—In this example, the *difference* of the co-efficients of x^2 is, $(a-b)$, and the difference of the co-efficients of y^2 is $(a-c)$. Hence, the answer is, $(a-b)x^2+(a-c)y^2$. See Art. 25.

On this principle polynomials may be made to undergo several useful transformations. Thus, $a^2-2ab+b^2$ may be written $a^2+b^2+(-2ab)$; $a^3-3a^2b+3ab^2-b^3$ may be written $a^3-b^3-(3a^2b-3ab^2)=a^3-b^3+3ab^2-(3a^2b)=a^3-(b^3-3ab^2+3a^2b)$.

MULTIPLICATION.

(35.) THE object of multiplication is to repeat one quantity as many times as there are units in the other.

If it is required to multiply a by b, the product may be written ab. (Art. 21.) If $a=5$, $b=4$, then the expression ab, which represents the product, becomes $5\times4=20$. It is obvious that the product of any number of factors will be the same when they are arranged in one order as when they are arranged in any other order. Thus, ba is the same as ab, and 4×5 is the same as 5×4. In the multiplication of literal quantities, it is customary to arrange the letters in alphabetical order.

(36.) In the multiplication of algebraical quantities, it is necessary to determine whether the *positive* or *negative* sign ought to be prefixed to the product of two monomial factors, or factors consisting of a single term. In order to do this, the following propositions must be established.

PROPOSITION I.

A positive quantity multiplied by a positive quantity gives a positive product.

Let it be required to multiply $+a$ by $+b$. Now, a is to be represented as many times as there are units in the multiplier, and as the sum of any number of positive quantities is positive, it follows that the product, ab, is positive. Hence the proposition is true.

PROPOSITION II.

If a negative quantity be multiplied by a positive quantity, the product will be negative; and, if a positive quantity be multiplied by a negative quantity, the product will be negative.

For, in the first place, let it be required to multiply $-a$ by $+b$. Now, we must repeat $-a$ as many times as there are units in b, and as the sum of any number of negative quantities is negative, it follows that the product of $-a$ and $+b$ is $-ab$. If $b=5$, then $-a$ must be repeated five times, and the product will be, in this case, $-a-a-a-a-a=-5a$.

In the second place, let it be required to multiply $+a$ by $-b$. In this case, $-b$ is to be repeated as many times as there are units in a, and as the sum of any number of negative quantities is negative, it follows that the product of $+a$ and $-b$ is $-ab$. Hence the proposition is true.

PROPOSITION III.

If a negative quantity be multiplied by a negative quantity, the product will be positive.

For, if $+a$ be multiplied by $+b$, the product is $+ab$. (Prop. I.) And, if $+a$ be multiplied by $-b$, the product is $-ab$. Hence we see that if a quantity be multiplied by a negative quantity, the sign of the product will be contrary to the sign of the product which was obtained by multiplying the *same* quantity by the *same* multiplier with a *positive* sign. Now, $-a$ multiplied by $+b$ gives $-ab$ for the product; whence, $-a$ multiplied by $-b$ gives $+ab$ for the product. Hence the proposition is true.

This proposition may be established in this manner: let it be required to multiply $a-b$ by $c-d$. In the first place, multiply $a-b$ by c, and we obtain for the product $ac-bc$. But as it is required to take $a-b$ as many as there are units in $c-d$, it follows that $a-b$ has been taken d times too often. Hence, if we multiply $a-b$ by d, and *subtract* the product from the first product, the true product will be obtained. The prod-

$$\left\{\begin{array}{l} a-b \\ c \\ \hline ac-bc \end{array}\right.$$

$$\left\{\begin{array}{l} a-b \\ d \\ \hline ad-bd \end{array}\right.$$

uct of d and $a-b$ is $ad-bd$; subtract this from $ac-bc$, by changing its signs, and we obtain for the product of $a-b$ by $c-d$, $ac-bc-ad+bd$. Hence, $-b$ multiplied by $-d$ gives $+bd$ for the product. If, in this example, we make $a=10$, $b=6$, $c=8$, $d=5$, the product, $ac-bc-ad+bd$ becomes $10\times8-6\times8-10\times5+6\times5=80-48-50+30=12$. This is obviously the true product, for $a-b=10-6=4$; and $c-d=8-5=3$, and $4\times3=12$.

(**37.**) From the preceding propositions, the following rule for the signs is derived.

RULE.

The product of two quantities affected with the same sign, is positive, and the product of two quantities affected with different signs is negative.

Multiplication may be divided into three cases.

CASE I.

(**38.**) *When the multiplicand and the multiplier are both monomials.*

Let it be required to multiply $5a^2x$ by $4xy$. The product may be expressed thus, $5a^2x\times4xy$. Now, the value of this product will not be changed by writing the factors in a different order, and therefore $5a^2x\times4xy$ may be written $4\times5\times a^2x\times xy=20\times a^2x\times xy=20\times a^2xxy$, since literal quantities may be multiplied by writing the factors which compose them, one after another. (Art. 21.) In the last expression the factor x is found twice, and $x\times x=x^2$. (Art. 11.)

Powers of the *same* quantity are multiplied by adding their exponents; for $a^3=a\times a\times a$, and $a^2=a\times a$. (Art. 11.) Therefore $a^3\times a^2=(a\times a\times a)\times(a\times a)=aaaaa=a^5$. $a^m\times a^n=a^{m+n}$.

From what has been said, we derive the following rule for the multiplication of monomials.

RULE.

To the product of the co-efficients affix that of the letters.

EXAMPLES.

1. Multiply $5a^2x$ by $4ab$. *Ans.* $20a^3bx$.

2. Multiply $17a^3x^2$ by $4a^2x^3z$. *Ans.* $68a^5x^5z$.

3. Multiply $19ay^2$ by $7by^3$. *Ans.* $133aby^5$.

4. Multiply $7ax$ by $41a^2x^2y^2$. *Ans.* $287a^3x^3y^2$.

5. Multiply $16ab$ by $17ax$. *Ans.* $272a^2bx$.

6. Multiply $14ay^2$ by $14y^2z^2$. *Ans.* $196ay^4z^2$.

7. Multiply $17ah^2$ by $11ah^3$. *Ans.* $187a^2h^5$.

8. Multiply $-6a^2b$ by $14a^2b^2$. *Ans.* $-84a^4b^3$.

9. Multiply $-14c^2$ by $10c^2x$. *Ans.* $-140c^4x$.

10. Multiply $21ay$ by $21by^2$. *Ans.* $441aby^3$.

11. Multiply $15ay^3$ by $-17ax^2$. *Ans.* $-255a^2x^2y^3$.

12. Multiply $23c^2x$ by $40n^2y$. *Ans.* $920c^2n^2xy$.

CASE II.

(39.) *When the multiplicand is a polynomial, and the multiplier a monomial.*

In this case we have the following obvious

RULE.

Multiply each term in the multiplicand by the multiplier, and connect the partial products by their respective signs.

EXAMPLES.

Operation.

1. Multiply $4a^2-3b+c$ by $4a$.

$$\begin{array}{r} 4a^2-\ \ 3b+c \\ 4a \\ \hline \textit{Ans.}\ \ 16a^3-12ab+4ac. \end{array}$$

2. Multiply $7ax-ay+2c$ by $11a$.

Ans. $77a^2x-11a^2y+22ac$.

3. Multiply $17a^2-2ac+4$ by $25c$.

Ans. $425a^2c-50ac^2+100c$.

4. Multiply $a^2+2ab+b^2$ by $4b^2$. *Ans.* $4a^2b^2+8ab^3+4b^4$.

5. Multiply $2ax+5ax-4ac$ by $4c$. *Ans.* $28acx-16ac^2$.*

6. Multiply $a^2y-5ac+3ac$ by $14a$. *Ans.* $14a^3y-28a^2c$.

7. Multiply $a^2-2ab+b^2$ by $-4a$.

Ans. $-4a^3+8a^2b-4ab^2$.

8. Multiply $a^{2n}-2a^nb^n+b^{2n}$ by $4a^{2n}$.

Ans. $4a^{4n}-8a^{3n}b^n+4a^{2n}b^{2n}$.

9. Multiply $a^3+3a^2b+3ab^2+b^3$ by $8a^3$.

Ans. $8a^6+24a^5b+24a^4b^2+8a^3b^3$.

10. Multiply $-4x^2+7xy-4ac$ by $-4a^2$.

Ans. $16a^2x^2-28a^2xy+16a^3c$.

CASE III.

(40.) *When both multiplicand and multiplier are polynomials.*

Let it be required to multiply $a+b$ by $c+b$. Now, we must repeat $a+b$ as many times as there are units in $c+b$. Hence, $a+b$ must be multiplied by c and then by b, and the two partial products added.

Operation.

$$\begin{array}{l} a+b \\ \underline{c+b} \\ ac+bc \\ \underline{\qquad\quad +ab+b^2} \\ ac+bc+ab+b^2 \text{ product.} \end{array}$$

---

* Note.—In this example the similar terms $2ax$ and $5ax$ may be added before performing the multiplication.

If the product of two polynomials contains similar terms, these must be arranged under each other in performing the multiplication, and their sum taken, as in the following example: Multiply $3a+b$ by $7a-2b$.

Operation.

$$\begin{array}{l} 3a+\ b \\ 7a\ -2b \\ \hline 21a^2+7ab \\ \qquad\ \ -6ab-2b^2 \\ \hline 21a^2+\ ab-2b^2 \text{ ans.} \end{array}$$

In this example, we first multiply $3a+b$ by $7a$, the first, or the left hand term, in the multiplier, and obtain for the product, $21a^2+7ab$; then we multiply $3a+b$ by the second term of the multiplier, $-2b$, and obtain for the product $-6ab-2b^2$. If now, these two partial products be added, by placing the similar terms, $7ab$, $-6ab$, under each other, the entire product, $21a^2+ab-2b^2$, is obtained.

From the foregoing examples and illustrations, we derive the following

RULE.

Multiply each term in the multiplicand by each term in the multiplier, in succession, and add the partial products for the entire product.

For the sake of system, it is better to multiply first by the left hand term of the multiplier, and then by each one in order.

1. Multiply $a^2+2ab+b^2$ by $a+b$.

Operation.

$$\begin{array}{llll} a^2+2ab & +b^2 & & \\ a\ +b & & & \\ \hline a^3+2a^2b+ & ab^2 & & =\text{the product by } a. \\ \quad\ \ a^2b+ & 2ab^2 & +b^3 & =\text{the product by } b. \\ \hline a^3+3a^2b+ & 3ab^2 & +b^3 & =\text{the entire product.} \end{array}$$

2. Multiply x^2+y^2 by x^2-y^2. *Ans.* x^4-y^4.

3. Multiply $x^2+2xy+y^2$ by $x-y$.
Ans. $x^3+x^2y-xy^2-y^3$.

4. Multiply $5a^4-2a^3b+4a^2b^2$ by $a^3-4a^2b+2b^3$.
Ans. $5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$.

5. Multiply $x^4+2x^3+3x^2+2x+1$ by x^2-2x+1.
Ans. x^6-2x^3+1.

6. Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$.
Ans. $a^4-2a^2b^2+b^4=(a^2-b^2)^2$.

7. Multiply x^2+xy+y^2 by x^2-xy+y^2.
Ans. $x^4+x^2y^2+y^4=(x^2+y^2)^2-x^2y^2$.

8. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Ans.* x^4-y^4.

9. Multiply $x^2+y^2+z^2-xy-xz-yz$ by $x+y+z$.
Ans. $x^3+y^3+z^3-3xyz$.

10. Multiply $a^2+2ab+b^2$ by a^2-b^2.
Ans. $a^4+2a^3b-2ab^3-b^4$.

11. Multiply $ab+cd$ by $ab-cd$. *Ans.* $a^2b^2-c^2d^2$.

12. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^6-3a^4b^2+3a^2b^4-b^6$.

13. Multiply $6-3+4$ by $9-5$.
Ans. $54-27+36-30+15-20$.

14. Multiply $1-x+x^2-x^3$ by $1+x$. *Ans.* $1-x^4$.

15. Multiply a^m+b^m by a^n+b^n.
Ans. $a^{m+n}+a^nb^m+a^mb^n+b^{m+n}$.

16. Multiply $a+b+c$ by $a+b+c$.
Ans. $a^2+b^2+c^2+2ab+2ac+2bc$.

17. Multiply $a+b+c$ by $a+b-c$.
Ans. $a^2+2ab+b^2-c^2$.

18. Multiply $a+b+c$ by $a-b-c$.
Ans. $a^2-b^2-2bc-c^2$.

19. Multiply $4xy+y^2$ by $4y^2+1$.

Ans. $16xy^3+4y^4+4xy+y^2$.

20. Multiply $3a^2b^2+2a^3b+4ab^3+4a^4+b^4$ by $ab+a^2$.*

Ans. $4a^6+6a^5b+5a^4b^2+7a^3b^3+5a^2b^4+ab^5$.

21. Multiply $4x^3-3x^2+x$ by $1+x$.

Ans. $4x^4+x^3-2x^2+x$.

22. Multiply $x-5$ by $x-7$. *Ans.* $x^2-12x+35$.

23. Multiply together $(x-a)$, $(x-b)$, $(x-c)$.

Ans. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

24. Multiply $a^{\frac{2}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{2}{3}}+b^{\frac{2}{3}}$.

Ans. $a^{\frac{4}{3}}+2a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$.

(41.) We will now establish some useful theorems.

THEOREM I.

The square of the sum of two quantities is equal to the square of the first, plus twice their product, plus the square of the second.

For, let a=one of the quantities, and b=the other. Then, their sum is $a+b$, and the square of their sum is, $(a+b)\times(a+b)$ $=a^2+2ab+b^2$. Hence the *theorem* is true.

THEOREM II.

The square of the differences of two quantities is equal to the square of the first, minus twice their product, plus the square of the second.

---

* NOTE.—In the preceding examples both multiplicand and multiplier have been arranged with reference to the powers of some letter; but in this example they are not so arranged. Before performing the multiplication, arrange them with reference to the powers of the letter a, that is, place the term which contains the highest power of a for the first, or left hand term, and the term which contains the next highest power of a for the second term, &c.

For, let a=one of the quantities, and b=the other. Then, $a-b$=their difference, and the square of their difference is, $(a-b)\times(a-b)=a^2-2ab+b^2$. Hence, the theorem is true.

THEOREM III.

The product of the sum and difference of two quantities is equal to the difference of their squares.

For, let a=one of the quantities, and b=the other. Then, $a+b$=their sum, and $a-b$=their difference. The product of their sum and difference is, $(a+b)\times(a-b)=a^2-b^2$. Hence, the theorem is true.

DIVISION.

(42.) Division has for its object the finding of one of two factors, when their product and one of the factors are given. The product of the two factors is the dividend, the given factor is the divisor, and the required factor is the quotient.

Since the dividend is equal to the product of the divisor by the quotient, it follows from what has been said in regard to the signs in Art. 37, that when the signs of the divisor and dividend are alike, the quotient must be positive, and when they are unlike the quotient must be negative. Thus, $+6\div-2$, or $-6\div+2$ is equal to -3, because the signs are *unlike.* But, $+6\div+2$, or $-6\div-2$, is equal to $+3$, because the signs are *alike.* We have, then, to determine the sign of the quotient, this simple

RULE.

When the signs of the divisor and dividend are alike, the quotient will be positive, and when they are unlike, the quotient will be negative.

(43.) We have seen by Art. 38, that powers of the same quantity may be multiplied by *adding* their exponents. Now, division is the reverse of multiplication; hence, powers of the same quantity may be divided by *subtracting* their exponents. Thus, $a^5 \div a^3 = a^2$, and $a^4 \div a^4 = a^0 = 1$, since a quantity divided by itself is equal to unity. *Hence, we see that a quantity which has a cipher for its exponent is equal to unity.*

If we divide a^4 by a^7, by subtracting the exponent of the divisor from that of the dividend, we obtain for a quotient a^{-3}. But $a^4 \div a^7 = \frac{a^4}{a^7} = \frac{1}{a^3}$; whence, $a^{-3} = \frac{1}{a^3}$, the reciprocal of a^3. Similarly, $a^{-5} = \frac{1}{a^5}$, the reciprocal of a^5. In general, *any quantity with a negative exponent, is equal to the reciprocal of that quantity with an equal positive exponent.*

Division may be divided into three cases.

CASE I.

When the divisor and dividend are monomials.

Let it be required to divide $24a^3b^2c$ by $4a^2b^3$. Now, the quotient multiplied by the divisor must equal the dividend. Therefore, by the principles of multiplication, the quotient must be $6ab^{-1}c$. This quotient may be obtained by dividing the co-efficient of the dividend by the co-efficient of the divisor, and the literal factors in the quotient are the same as those in the dividend with their exponents diminished by the exponents of the corresponding letters in the divisor. In this example, the factor c, which is found in the dividend, is not found in the divisor, and therefore the exponent of c in the divisor is nothing, since by placing the factor c^0 in the divisor we do not change its value. (Art. 43.) Hence, there is nothing to subtract from the exponent of c in the dividend, which is understood, and c must be found in the quotient, as a factor.

If, in the example above, the letter c had been found in the divisor instead of the dividend, it is plain, from what has been said, that the exponent of c in the quotient must have been -1.

Hence, we have the following rule for dividing one monomial by another.

RULE.

Divide the co-efficient of the dividend by the co-efficient of the divisor, and to the quotient affix the letters of the dividend with the exponent of each diminished by the exponent of the same letter in the divisor.

It must be observed, that in some examples, the same letters are not found in both dividend and divisor, but the dividend and divisor may always be made to contain the same letters, and then the rule may be applied. Thus, let it be required to divide $35a^3b^2$ by $7abc^2$. By Art. 43, the dividend may be written $35a^3b^2c^0$ without changing its value. If the rule be now applied, we obtain for the quotient, $5a^2bc^{-2}$.

Whenever the exponent of any quantity in the quotient is a cipher, that quantity may be omitted, since any quantity which has a cipher for its exponent is equal to unity. (Art. 43.)

EXAMPLES.

1. Divide $49a^2b^2$ by $7abc^2$. *Ans.* $7abc^{-2}$.

2. Divide $-72a^3b^4c$ by $9ab^2$. *Ans.* $-8a^2b^2c$.

3. Divide $64x^3y^2$ by $16x^2y^4$. *Ans.* $4xy^{-2}$.

4. Divide $625a^2b^7$ by $25a^3b^5$. *Ans.* $25a^{-1}b^2$.

5. Divide $484a^3m^2$ by $22am^4$. *Ans.* $22a^2\dfrac{1}{m^2}$.

6. Divide $73c^3h^3$ by $11chn^2$.* *Ans.* $\frac{73}{11}c^2h^2n^{-2}$.

* Note.—In this example, 11 is not contained in 73, and the division is expressed by writing 11 under 73, in the form of a fraction. The same may be done in any similar case, and the fraction may be reduced to its lowest terms, as in arithmetic.

7. Divide $-48a^3b^7$ by $-24a^2b^4$. *Ans.* $2ab^3$.

8. Divide $96a^7b^9$ by $-72a^5b^7$. *Ans.* $-\frac{4}{3}a^2b^2$.

9. Divide $252e^9k^3$ by $364eky^2$. *Ans.* $\frac{9}{13}e^8k^2y^{-2}$.

10. Divide $144m^5n^2$ by $240m^4n^4$. *Ans.* $\frac{3}{5}mn^{-2}$.

11. Divide $192m^{12}n^3$ by $1536m^8n^5$. *Ans.* $\frac{1}{8}m^4.\frac{1}{n^2}$.

12. Divide $246a^{11}b^2c^3$ by $372a^9bc$. *Ans.* $\frac{41}{62}a^2bc^2$.

CASE II.

(45.) *When the dividend is a polynomial and the divisor a monomial.*

Let it be required to divide $3a^2b-6ab$ by $3a$, or, in other words, to find a quantity which multiplied by $3a$, the divisor, will give $3a^2b-6ab$. Now, since each term in the required multiplicand, must be *repeated* $3a$ times, it follows that if each term of the quantity, $3a^2b-6ab$, be *divided* by $3a$, the required quantity will be found. Hence, to divide a polynomial by a monomial, we have the following

RULE.

Divide each term in the dividend by the divisor, and the sum of these partial quotients will be the quotient required.

EXAMPLES.

1. Divide $3a^3b^2-18a^4b^2+6a^2b$ by $3ab$.
Ans. a^2b-6a^3b+2a.

2. Divide $24x^2y^2-8x^4y^5-24xy^2$ by $8x$.
Ans. $3xy^2-x^3y^5-3y^2$.

3. Divide $21a^3x^3-7a^2x^2+14ax$ by $7ax$.
Ans. $3a^2x^2-ax+2$.

4. Divide $42a^3-11a^2+28a$ by $7a$.
Ans. $6a^2-\frac{11}{7}a+4$.

5. Divide $9k^{14}-24k^3c+48k^5$ by $3k^4$.

Ans. $3k^{10}-8k^{-1}c+16k$.

6. Divide $72a^5c^2-48a^7c^4-32a^9c^5$ by $16a^3c^3$.

Ans. $\frac{9}{2}a^2c^{-1}-3a^4c-2a^6c^2$.

7. Divide $36m^{\frac{7}{2}}-48m^{\frac{3}{4}}$ by $4m^{\frac{1}{3}}$.

Ans. $9m^{\frac{19}{6}}-12m^{\frac{5}{12}}$.

8. Divide $m^{\frac{3}{4}}-m^{\frac{3}{5}}n^{\frac{2}{3}}$ by $m^{\frac{1}{4}}$. *Ans.* $m^{\frac{1}{2}}-m^{\frac{7}{20}}n^{\frac{2}{3}}$.

9. Divide $a^{\frac{5}{6}}-a^{\frac{1}{6}}b^2+a^{\frac{4}{6}}$ by $a^{\frac{1}{6}}$. *Ans.* $a^{\frac{2}{3}}-b^2+a^{\frac{1}{2}}$.

10. Divide $11a^{\frac{5}{9}}-33a^{\frac{4}{3}}$ by $11a^{\frac{2}{9}}$. *Ans.* $a^{\frac{1}{3}}-3a^{\frac{10}{9}}$.

11. Divide $72m^{\frac{10}{12}}-60m^{\frac{1}{6}}n^{\frac{2}{3}}$ by $24m^{\frac{1}{6}}$. *Ans.* $3m^{\frac{2}{3}}-\frac{5}{2}n^{\frac{2}{3}}$.

12. Divide $2a^4+4a^3b+2a^2b^2$ by $2a^2$. *Ans.* $a^2+2ab+b^2$.

CASE III.

(46.) *When both the dividend and divisor are polynomials.*

In order to give greater simplicity to the division of one polynomial by another, it will be found necessary to arrange both dividend and divisor according to the powers of the same letter. It will be understood, therefore, that in all cases they must be so arranged before performing the division.

Let it be required to divide $2a^5+12a^4+22a^3+12a^2$ by a^2+3a. In this example, both dividend and divisor are already arranged with reference to the powers of the letter a, and the dividend being the product of the divisor and quotient, it follows that the quotient will be arranged with reference to the powers of the letter a Since the dividend is the sum of all the partial products that can be formed by *multiplying* each term of the divisor by each term of the quotient, it follows that the first term of the quotient is found by *dividing* the first term of the dividend by the first term of the divisor. Therefore the first term in the quotient is, $2a^5\div a^2=2a^3$. If the divisor be now multiplied by the first term in the

quotient, $2a^3$, and the product, $2a^5+6a^4$, be subtracted from the dividend, as represented in the following operation, the remainder, $6a^4+22a^3+12a^2$, must be the sum of all the partial products that can be found by multiplying each term of the divisor by each of the remaining terms of the quotient. Therefore, the first remainder may be regarded as being a new dividend, and the second term of the quotient may be found in the same manner as the first term of the quotient was obtained, by dividing the first term of the new dividend by the first term of the divisor. Hence, $6a^4 \div a^2 = 6a^2$, the second term of the quotient. Multiply the divisor by $6a^2$, and subtract the product, $6a^4+18a^3$, from the first remainder, and the second remainder is, $4a^3+12a^2$. From what has already been stated, $4a^3 \div a^2 = 4a$, the third term in the quotient. Multiply the divisor by $4a$, and subtract the product, $4a^3+12a^2$, from the second remainder, and there is nothing remaining. Hence, $(2a^5+12a^4+22a^3+12a^2) \div (a^2+3a) = 2a^3+6a^2+4a$.

Operation.

$$
\begin{array}{r|l}
2a^5+12a^4+22a^3+12a^2 & a^2+3a \qquad =\text{divisor.} \\
\cline{2-2}
2a^5+6a^4 & 2a^3+6a^2+4a=\text{quotient.} \\
\cline{1-1}
\text{1st remainder}= \quad 6a^4+22a^3+12a^2 & \\
6a^4+18a^3 & \\
\cline{1-1}
\text{2d remainder}= \qquad\qquad 4a^3+12a^2 & \\
4a^3+12a^2 & \\
\cline{1-1}
\text{3d remainder}= \qquad\qquad\qquad 0 \quad & \\
\end{array}
$$

From what has been said, we derive the following rule for the division of one polynomial by another.

RULE.

1. *Arrange the divisor and dividend with reference to the powers of the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor, and the result will be the first term of the quotient, by which multiply the divisor, and subtract the product from the dividend.*

3. *Divide the first term of the remainder by the first term of the divisor, and the result will be the second term in the quotient, by which multiply the divisor, and subtract the product from the first remainder, and proceed as before to find the remaining terms of the quotient.*

EXAMPLES.

1. Divide $a^3+3a^2b+3ab^2+b^3$ by $a+b$.

Operation.

$$
\begin{array}{l|l}
a^3+3a^2b+3ab^2+b^3 & a+b \qquad =\text{divisor.} \\ \cline{2-2}
\underline{a^3+\ a^2b} & a^2+2ab+b^2=\text{quotient.} \\ \cline{2-2}
\qquad 2a^2b+3ab^2{*} & \\
\qquad \underline{2a^2b+2ab^2} & \\
\qquad\qquad\quad ab^2+b^3 & \\
\qquad\qquad\quad \underline{ab^2+b^3} &
\end{array}
$$

2. Divide $6a^4-96$ by $3a-6$.

Operation.

$$
\begin{array}{l|l}
6a^4-96 & 3a-6 \qquad\qquad =\text{divisor.} \\ \cline{2-2}
\underline{6a^4-12a^3} & 2a^3+4a^2+8a+16=\text{quotient.} \\
\qquad 12a^3-96 & \\
\qquad \underline{12a^3-24a^2} & \\
\qquad\qquad 24a^2-96 & \\
\qquad\qquad \underline{24a^2-48a} & \\
\qquad\qquad\qquad 48a-96 & \\
\qquad\qquad\qquad \underline{48a-96} & \\
\qquad\qquad\qquad\qquad 0 &
\end{array}
$$

3. Divide $a^2-2ab+b^2$ by $a-b$. *Ans.* $a-b$.

4. Divide $12x^4-192$ by $3x-6$.†

Ans. $4x^3+8x^2+16x+32$.

* NOTE.—It will be observed that it is unnecessary to bring down the whole remainder, $2a^2b+3ab^2+b^3$.

† NOTE.—In this example, and in many others, the dividend and divisor may be divided by a common factor before performing the division.

5. Divide $6x^6-6y^6$ by $2x^2-2y^2$.

Ans. $3x^4+3x^2y^2+3y^4$.

6. Divide $x^3+5x^2y+5xy^2+y^3$ by $x^2+4xy+y^2$.

Ans. $x+y$.

7. Divide a^4-b^4 by $a-b$. *Ans.* $a^3+a^2b+ab^2+b^3$.

8. Divide $x^3-9x^2+27x-27$ by $x-3$.

Ans. x^2-6x+9.

9. Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.

Ans. $24x^2-2ax-35a^2$.

10. Divide $\frac{1}{2}x^3+x^2+\frac{3}{8}x+\frac{3}{4}$ by $\frac{1}{2}x+1$. *Ans.* $x^2+\frac{3}{4}$.

11. Divide x^4+y^4 by $x+y$.

Ans. $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$.

12. Divide x^5-y^5 by $x-y$.

Ans. $x^4+x^3y+x^2y^2+xy^3+y^4$.

13. Divide $a^{2m}+2a^mb^n+b^{2n}$ by a^m+b^n.

Ans. a^m+b^n.

14. Divide $x^4+x^2y^2+y^4$ by x^2-xy+y^2.

Ans. x^2+xy+y^2.

15. Divide x^5+y^5 by $x+y$.

Ans. $x^4-x^3y+x^2y^2-xy^3+y^4$.

16. Divide $6c^4+9c^2-15c$ by $3c^2-3c$.

Ans. $2c^2+2c+5$.

17. Divide $x^6-x^4+x^3-x^2+2x-1$ by x^2+x-1.

Ans. $x^4-x^3+x^2-x+1$.

18. Divide c^6-y^6 by $c^3+2c^2y+2cy^2+y^3$.

Ans. $c^3-2c^2y+2cy^2-y^3$.

Both dividend and divisor may also be multiplied by the same quantity before dividing.

19. Divide y^3-1 by $y-1$. *Ans.* y^2+y+1.

20. Divide x^3+y^3 by $x+y$. *Ans.* x^2-xy+y^2.

21. Divide $1+x$ by $1-x$.*

Ans. $1+2x+2x^2+2x^3+\dfrac{2x^4}{1-x}$.

22. Divide 1 by $1-2x+x^2$.

Ans. $1+2x+3x^2+4x^3+$a fraction.

23. Divide $2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}$ by a^n-b^n.

Ans. $2a^{2n}-4a^nb^n+2b^{2n}$.

24. Divide a^2-b^2 by $a-b$. *Ans.* $a+b$.

PROPOSITION.

(**47.**) *The difference of the same powers of two quantities is divisible by their difference.*

Let a^n and b^n represent two quantities whose common exponent is n. Then a^n-b^n is divisible by $a-b$.

By commencing the division we have,

$$\begin{array}{l|l} a^n-\quad b^n & a-b \text{ divisor.} \\ a^n-a^{n-1}b & a^{n-1} \text{ the first term of the quotient.} \\ \hline \end{array}$$

$$\text{Remainder}= \quad a^{n-1}b-b^n=b(a^{n-1}-b^{n-1}).$$

Now since the dividend is equal to the quotient multiplied by the divisor plus the remainder, it follows that,

$$a^n-b^n=(a-b)a^{n-1}+b(a^{n-1}-b^{n-1}) \qquad \text{(A)}.$$

In equation (A), the term $(a-b)a^{n-1}$ is obviously divisible by $a-b$, the quotient being a^{n-1}. Therefore, whenever the term, $b(a^{n-1}-b^{n-1})$, or $a^{n-1}-b^{n-1}$, is divisible by $a-b$, the right hand

---

* NOTE.—In this example the division does not terminate. In such cases, we can write the divisor under the last remainder, in the form of a fraction, and add this fraction to that part of the quotient which is obtained.

member of equation (A) is divisible by $a-b$, and consequently its equal, a^n-b^n, is also divisible by $a-b$. Hence, if the difference of the same powers of two quantities is divisible by the difference of those quantities, then the difference of the powers of the quantities will be divisible by their difference, when their exponents are increased by unity. Now, $\overline{a^2-b^2}\div\overline{a-b}=a+b$. Therefore, (a^3-b^3) is divisible by $(a-b)$, and in general a^m-b^m is divisible by $a-b$.

USEFUL FORMULAS.

1. $(a^2-b^2)=(a+b)(a-b)$.

2. $(a^4-b^4)=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b)$.

3. $(a^3+b^3)=(a^2-ab+b^2)(a+b)$.

4. $(a^3-b^3)=(a^2+ab+b^2)(a-b)$.

5. $(a^6-b^6)=(a^3+b^3)(a^3-b^3)=(a^3+b^3)(a^2+ab+b^2)(a-b)$.

6. $(a^6-b^6)=(a^3+b^3)(a^3-b^3)=(a^3-b^3)(a^2-ab+b^2)(a+b)$.

7. $(a^6-b^6)=(a^3+b^3)(a^3-b^3)=(a^2-b^2)(a^4+a^2b^2+b^4)$.

8. $(a^6-b^6)=(a^3+b^3)(a^3-b^3)=(a-b)(a^2+ab+b^2)(a^2-ab+b^2)(a+b)$.

9. $(a^2-b^2)\div(a+b)=a-b$.

10. $(a^3+b^3)\div(a+b)=a^2-ab+b^2$.

11. $(a^3-b^3)\div(a-b)=a^2+ab+b^2$.

12. $(a^4-b^4)\div(a^2-b^2)=(a^2+b^2)$.

13. $(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$.

14. $(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4$.

15. $(a^6-b^6)\div(a^2-b^2)=a^4+a^2b^2+b^4$.

16. $a^0=1$.

THE GREATEST COMMON MEASURE.

(48.) *The greatest common measure of two or more quantities is the greatest factor which is common to all of them.* Thus, the greatest common measure of $49a^2x^3$ and $35a^2x^2$ is $7a^2x^2$. In examples like this, the greatest common measure may be found by inspection, but in many others it will be necessary to apply the following rule for finding the common measure of two polynomials.

RULE.

Arrange the two polynomials with reference to the powers of some letter, and divide that which contains the highest power of this letter, by the other; then divide the last divisor by the last remainder, and continue this process till there is no remainder; the last divisor will be the greatest common divisor.

(49.) When the first term of any dividend is not divisible by the first term of the divisor, it may be made so by multiplying each term of the dividend by any quantity which will render it divisible. The reason of this is obvious.

(50.) If either the dividend or divisor contain any simple factor which is not common to both, this factor may be rejected, before commencing the division, as this factor can form no part of the greatest common measure.

(51.) If both polynomials contain a simple common measure, this common measure may be rejected from each before applying the rule; but as it forms a part of the greatest common divisor, it must be restored in the last divisor.

DEMONSTRATION OF THE RULE.

Let P and p be any two polynomials, of which P is the greater, or the one which contains the highest power of that letter, with reference to which the two polynomials are arranged. If we now operate on these two polynomials, according to the rule, we shall have the following operation:

$$
\begin{array}{llll}
 & p)\ P\ (q_1{}^* & & =\text{first quotient.} \\
 & \underline{pq_1} & & \\
\text{1st remainder}= & r_1)\ p\ (q_2 & & =\text{second quotient.} \\
 & \quad \underline{r_1q_2} & & \\
\text{2d remainder}= & \quad r_2)\ r_1\ (q_3 & & =\text{third quotient.} \\
 & \qquad \underline{r_2q_3} & & \\
\text{3d remainder}= & \qquad 0 & &
\end{array}
$$

Now, since the dividend is equal to the divisor multiplied by the quotient plus the remainder, we have the following equations:

$$P=pq_1+r_1 \quad (1)$$
$$p=r_1q_2+r_2 \quad (2)$$
$$r_1=r_2q_3+0 \quad (3)$$

Since r_2 measures the right hand member of equation (3) it must also measure its left hand member, r_1. Therefore, r_2 must measure each member of equation (2). Again, since r_2 measures r_1 and p, it must measure each member of equation (1), and therefore measures P. Whence, r_2 is a common measure of P and p. We say, further, that it is their greatest common measure. For, equations (1) and (2) may be written as follows:

$$P-pq_1=r_1 \quad (1)'$$
$$p-r_1q_2=r_2 \quad (2)'$$

Now, every common measure of P and p must measure each member of equation (1)$'$, and consequently each member of equation (2)$'$. But it has been found that r_2 is a common measure of P and p, and therefore r_2 is a common measure of each member of equation (2)$'$. The greatest common measure of r_2 is *itself;* therefore the greatest common measure of P and p is r_2. Hence the rule is established.

EXAMPLES.

1. What is the greatest common measure of $a^2-ab-2b^2$ and $a^2-3ab+2b^2$?

---

* NOTE.—The numerals which are placed at the bottom of the letters q and r, are used to avoid the necessity of introducing too many letters into any calculation. They have other advantages which the pupil will soon learn to perceive. q_1 is read q sub one, q_2, q sub two, &c. q_1 q_2, &c., represent different quantities.

Operation.

$$
\begin{array}{lll}
 & a^2-\ ab-2b^2 & \big|\ a^2-3ab+2b^2\text{, divisor.} \\
 & \underline{a^2-3ab+2b^2} & \big|\ 1 \\
\text{1st remainder} & 2b)\overline{2ab-4b^2} & \\
 & \qquad a\ -2b)a^2-3ab+2b^2(a-b & \\
 & \qquad\qquad \underline{a^2-2ab} & \\
 & \qquad\qquad\quad -ab+2b^2 & \\
 & \qquad\qquad\quad \underline{-ab+2b^2} & \\
 & \qquad\qquad\qquad\quad 0 &
\end{array}
$$

In this example we first divide $a^2-ab-2b^2$ by $a^2-3ab+2b^2$ and find that it is contained once, with $2ab-4b^2$ for a remainder By examining this remainder it may be seen that $2b$ is a factor of it, but that it is not a factor of the divisor, $a^2-3ab+2b^2$. Therefore, by Art. 50, divide $2ab-4b^2$ by $2b$, and we have $a-2b$, which is contained in the divisor $a^2-3ab+2b^2$, $a-b$ times, with no remainder. Therefore, $a-2b$ is the greatest common divisor required.

2. What is the greatest common measure of $12a^2-8a-4$, and $20a^3c-10a^2c-15ac+5c$?

Divide the first by 4, and the second by $5c$, and we have

$$
\begin{array}{ll}
 & 3a^2-2a-\ 1 \\
\text{and} & 4a^3-2a^2-\ 3a+\ 1
\end{array}
$$

$$
\begin{array}{lll}
\text{Multiply by 3 (Art. 49)} & \underline{3} & \\
 & 12a^3-6a^2-\ 9a+\ 3 & \big|\ 3a^2-2a-1\text{ divisor.} \\
 & \underline{12a^3-8a^2-\ 4a} & \big|\ 4a\ +2 \\
 & \qquad 2a^2-\ 5a+\ 3 & \\
\text{Multiply by 3} & \qquad \underline{3} & \\
 & \qquad 6a^2-15a+\ 9 & \\
 & \qquad \underline{6a^2-\ 4a-\ 2} & \\
\text{Divide by }-11\text{ (Art. 50)} & -11)\overline{-11a+11} & \\
 & \qquad\qquad a-\ 1)3a^2-2a-1(3a+1 & \\
 & \qquad\qquad\qquad \underline{3a^2-3a} & \\
 & \qquad\qquad\qquad\qquad a-1 & \\
 & \qquad\qquad\qquad\qquad \underline{a-1} & \\
 & \qquad\qquad\qquad\qquad\quad 0 &
\end{array}
$$

Therefore $a-1$ is the greatest common measure.

3. What is the greatest common measure of x^3-xy^2 and $x^2+2xy+y^2$? *Ans.* $x+y$.

4. What is the greatest common measure of $x^4-8x^3+21x^2-20x+4$ and $2x^3-12x^2+21x-10$. *Ans.* $x-2$.

5. What is the greatest common measure of $x^3-7x^2+14x-8$ and $3x^4-14x^3+8x^2$? *Ans.* $x-4$.

6. What is the greatest common measure of $x^3+4ax^2+5a^2x+2a^3$ and $3x^2+7ax+2a^2$? *Ans.* $x+2a$.

7. What is the greatest common measure of $5x^4+13x^3+2x^2-8x$ and $2x^3+x^2+12$? *Ans.* $x+2$.

8. What is the greatest common measure of $x^3-7x^2+16x-12$ and $3x^2-14x+16$? *Ans.* $x-2$.

9. What is the greatest common measure of $4x^4-10x^3-6x^2+8x+4$ and $2x^3+5x^2+4x+1$? *Ans.* $2x+1$.

10. What is the greatest common measure of $12x^5-40x^3+60x+32$ and $3x^4-6x^2+3$? *Ans.* x^2+2x+1.

11. What is the greatest common measure of x^8-y^8 and $x^{13}-y^{13}$? *Ans.* $x-y$.

12. What is the greatest common measure of $a^3-a^2b+3ab^2-3b^3$ and $a^2-5ab+4b^2$? *Ans.* $a-b$.

13. What is the greatest common measure of x^3+y^3 and $x^2+2xy+y^2$? *Ans.* $x+y$.

14. What is the greatest common measure of a^4-b^4 and $a^3+a^2b-ab^2-b^3$? *Ans.* a^2-b^2.

(**52.**) In order to obtain the greatest common measure of three quantities, find the greatest common measure of two of them, then that of this common measure, and the third quantity will be the greatest common measure required. For, let a, b, c, be the three quantities, and m the greatest common measure of a and b, and n the greatest common measure of m and c. Since m is the greatest common measure of a and b, and n the greatest common

measure of m and c, n must measure a, b, and c. Therefore n, which is the greatest common measure of m and c, is also the greatest common measure of a, b, and c. This reasoning may be applied to any number of quantities.

15. What is the greatest common measure of a^4-b^4, $a^3-a^2b+3a-3b$, and $a^2+a-ab-b$?

In this example, first find the common measure of a^4-b^4 and $a^3-a^2b+3a-3b$.

$$a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b) \qquad (m)$$
$$a^3-a^2b+3a-3b \qquad =a^2(a-b)+3(a-b)=(a^2+3)(a-b)^*(n)$$

By examining the two polynomials (m) and (n), it is obvious that $a-b$ is their greatest common measure.

16. What is the common measure of $x^4+5x^3+6x^2$, x^3+3x^2+3x+2, and $3x^3+8x^2+5x+2$? *Ans.* $x+2$.

In this example, find the common measure of $x^4+5x^3+6x^2$ and x^3+3x^2+3x+2. Reject the factor x^2 from the first, before commencing the division.

17. What is the greatest common measure of $a^3+3a^2b+3ab^2+b^3$, $4a^2b^2+12ab^3+8b^4$ and a^6+b^6? *Ans.* $a+b$.

18. What is the greatest common measure of $5a^5+10a^4b+5a^3b^2$, $a^3b+2a^2b^2+2ab^3+b^4$, and a^3+b^3? *Ans.* $a+b$.

---

* NOTE.—Much of the pupil's success in algebra will depend on his ability to resolve algebraical expressions into their factors. To this end, he must carefully examine an algebraical quantity, in order to discover all the different forms in which it may be written, and thus cultivate a quickness of perception. It would be impossible to give rules for every operation in Algebra, and if we could, the labor would be unnecessary. Rules contain only general principles, and are chiefly designed to aid the beginner.

THE LEAST COMMON MULTIPLE.

(**53.**) A COMMON MULTIPLE of any quantity is a quantity which contains it without a remainder. A common multiple of two or more quantities, is a quantity which contains each of them without a remainder. *The least common multiple of two or more quantities is, therefore, the least quantity which contains each of them without a remainder.*

Let it be required to find the least common multiple of a^2x, cx, and abx. If these quantities be multiplied together, the product will obviously be their common multiple, but not their *least* common multiple; for the product is a^3bcx^2, which contains a^3, b, c, and x^2. But the highest powers of a and x, which are found in the three quantities, are a^2 and x, and therefore the least common multiple is a^2bcx. Hence, in finding the least common multiple of two or more quantities, we must observe that when different powers of the same factor occur in the given quantities, the least common multiple will contain only the highest power of this factor. The least common multiple of two or more quantities, then, may be found by either of the following rules.

RULE I.

Resolve each quantity into its prime factors, and then, if different powers of the same quantity occur as factors, reject all but the highest power of this quantity. Of the remaining factors, select all the different ones, and their product will be the least common multiple.

RULE II.

Arrange the quantities in a horizontal line, and then divide two or more of them by any prime factor which will measure them, and place the quotients and undivided terms in a horizontal line below. Proceed with this line as with the first, and so on, till all the quantities in the last line are prime with respect to each other. The continued product of the divisors and the quantities in the last line will be the least common multiple.

(54.) If no prime factor can be found by inspection which will measure two or more of the given quantities, the common divisors may be found by Art. 48 and Art. 52.

1. What is the least common multiple of $48a^2b^3$, $16a^4b^4$, $a^2+2ab+b^2$, and a^3+b^3?

SOLUTION BY RULE I.

$48a^2b^3=2^4\times3\times a^2\times b^3$,	the prime factors of $48a^2b^3$
$16a^4b^4=2^4\times a^4\times b^4$,	the prime factors of $16a^4b^4$
$a^2+2ab+b^2=(a+b)^2$	the prime factors of $a^2+2ab+b^2$
$a^3+b^3=(a+b)(a^2-ab+b^2)$	the prime factor of a^3+b^3

$\therefore$ $3\times2^4\times a^4\times b^4\times(a+b)^2(a^2-ab+b^2)=48a^4b^4\times(a+b)^2(a^2-ab+b^2)$ is the least common multiple required.

2. What is the least common multiple of $6a^2x^2(a-x)$, $8x^3(a^2-x^2)$, and $12(a-x)^2$?

SOLUTION BY RULE II.

$2x^2$	$6a^2x^2(a-x)$,	$8x^2(a^2-x^2)$,	$12(a-x)^2$
$a-x$	$3a^2(a-x)$,	$4(a^2-x^2)$,	$12(a-x)^2$
3	$3a^2$,	$4(a+x)$,	$12(a-x)$
4	a^2,	$4(a+x)$,	$4(a-x)$
	a^2,	$a+x$,	$a-x$

$\therefore$ $4\times3\times a^2\times2x^2\times(a-x)\times(a-x)\times(a+x)=24a^2x^2\times(a^2-x^2)\times(a-x)$, the least common multiple.

3. What is the least common multiple of $15a^2b^2$, $12ab^3$, and $6a^3b$? *Ans.* $60a^3b^3$.

4. What is the least common multiple of $(a+b)^2$, a^2-b^2, $(a-b)^2$, and $a^3+3a^2b+3ab^2+b^3$? *Ans.* $(a+b)(a^2-b^2)^2$.
Or, $(a+b)(a-b)^2(a+b)^2$.
Or, $(a+b)^3(a-b)^2$.

---

* NOTE.—By using $2x^2$ and 4 we do not rigidly follow the rule, because they are not prime factors, but it is easy to see that the result is the same as would have been obtained by dividing separately by the prime factors of $2x^2$ and 4.

5. What is the least common multiple of $x^2+2xy+y^2$ and x^3-xy^2? *Ans.* $x(x+y)(x^2-y^2)$.

6. What is the least common multiple of a^2-b^2 and a^3+b^3? *Ans.* $(a^2-b^2)(a^2-ab+b^2)$.

7. What is the least common multiple of y^2-8y+7, and y^2+7y-8? *Ans.* $y^3-57y+56$.

8. What is the least common multiple of $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2? *Ans.* a^6-b^6.

9. What is the least common multiple of $16ab$, $18a^2b^3$, $36ab^4x$, and $72abx^2$? *Ans.*

CHAPTER III.

FRACTIONS.

(**55.**) ALGEBRAICAL FRACTIONS do not differ in any respect from arithmetical fractions, and therefore the rules for operating upon them are the same as those in common arithmetic, and they are deduced in the same manner.

(**56.**) Since the value of a fraction is the quotient which is obtained by dividing the numerator by the denominator, we infer the following principles, upon which the principal rules are founded:

1. *That multiplying or dividing both numerator and denominator of a fraction by the same quantity does not change its value.*
2. *That multiplying the numerator, or dividing the denominator, of a fraction by any quantity, multiplies the fraction by that quantity.*
3. *That dividing the numerator, or multiplying the denominator, of any fraction by a quantity, divides the fraction by that quantity.*

CASE I.

(**57.**) *To reduce a fraction to its lowest terms.*

By the first principle in the preceding article, we can divide both numerator and denominator of a fraction by a quantity without changing its value. If both terms of the fraction be divided by their greatest common divisor, it is obvious that the fraction will then be reduced to its lowest terms; whence the following

RULE.

Divide both numerator and denominator by their greatest common measure.

EXAMPLES.

1. Reduce $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms.

By resolving both the numerator and denominator into factors, we readily discover the greatest common measure. Thus, $\frac{2x^3-16x-6}{3x^3-24x-9}=\frac{2\times(x^3-8x-3)}{3\times(x^3-8x-3)}=\frac{2}{3}$, the fraction required.

2. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

By applying the rule for finding the greatest common measure, we find that it is a^2-x^2. Divide both numerator and denominator by a^2-x^2, and the fraction becomes $\frac{a^2+x^2}{a-x}$.

3. Reduce the fraction $\frac{x^3+y^3}{x^2-y^2}$ to its lowest terms.

Ans. $\frac{x^2-xy+y^2}{x-y}$.

4. Reduce the fraction $\frac{a^3-b^3}{a-b}$ to its lowest terms.

Ans. $\frac{a^2+ab+b^2}{1}$.

5. Reduce the fraction $\frac{a^6-b^6}{a^4-b^4}$ to its lowest terms.

Ans. $\frac{a^4+a^2b^2+b^4}{a^2+b^2}$.

6. Reduce the fraction $\frac{24x^4-22x^2+5}{48x^4+16x^2-15}$ to its lowest terms.

Ans. $\frac{2x^2-1}{4x^2+3}$.

7. Reduce the fraction $\frac{x^2-a^2}{x^4-a^4}$ to its lowest terms.

Ans. $\frac{1}{x^2+a^2}$.

8. Reduce the fraction $\frac{20x^4+x^2-1}{25x^4+5x^3-x-10x^2+1}$ to its lowest terms.

Ans. $\frac{4x^2+1}{5x^2+x-1}$.

9. Reduce the fraction $\frac{a^3-b^3}{a^4-a^2b^2}$ to its lowest terms.

Ans. $\frac{a^2+ab+b^2}{a^2(a+b)}$.

10. Reduce the fraction $\frac{a^4-b^4}{a^3-a^2b-ab^2+b^3}$ to its lowest terms.

Ans. $\frac{a^2+b^2}{a-b}$.

11. Reduce the fraction $\frac{a^3-3a^2c+3ac^2-c^3}{a^2-c^2}$ to its lowest terms.

Ans. $\frac{a^2-2ac+c^2}{a+c}$.

12. Reduce the fraction $\frac{a^3-ac^2}{a^2+2ac+c^2}$ to its lowest terms.

Ans. $\frac{a^2-ac}{a+c}$.

CASE II.

(58.) To reduce a mixed quantity to the form of a fraction.

RULE.

Multiply the integral part by the denominator of the fraction, and to the product add the numerator, if the sign of the fraction is positive, but if the sign is negative subtract the numerator from the product. Under the result thus obtained, write the denominator.

The pupil will readily understand this rule, if he comprehends the corresponding rule in arithmetic.

EXAMPLES.

1. Reduce $a-\frac{a+ab}{a+b}$ to the form of a fraction.

$a\times(a+b)=a^2+ab$. The negative sign placed before the fraction shows that its numerator should be subtracted from this product. Hence, $\frac{a^2+ab-(a+ab)}{a+b}=\frac{a^2-a}{a+b}$ is the fraction required.

2. Reduce $1+\dfrac{a^2-x^2}{a^2+x^2}$ to the form of a fraction.

Ans. $\dfrac{2a^2}{a^2+x^2}$.

3. Reduce $1+\dfrac{b^2+c^2-a^2}{2bc}$ to the form of a fraction.

Ans. $\dfrac{(b+c)^2-a^2}{2bc}$.

4. Reduce $1-\dfrac{a^2-2ab+b^2}{a^2+b^2}$ to the form of a fraction.

Ans. $\dfrac{2ab}{a^2+b^2}$.

5. Reduce $1-\dfrac{b^2+c^2-a^2}{2bc}$ to the form of a fraction.

Ans. $\dfrac{a^2-(b-c)^2}{2bc}$.

6. Reduce $a+\dfrac{a^2+ab+b^2}{a+b}$ to the form of a fraction.

Ans. $\dfrac{2a^2+2ab+b^2}{a+b}$.

7. Reduce $5x-\dfrac{2x-5}{3}$ to the form of a fraction.

Ans. $\dfrac{13x+5}{3}$.

8. Reduce $a^2+2ab+b^2-\dfrac{a^2-2ab+b^2}{a+b}$ to the form of a fraction.

Ans. $\dfrac{(a+b)^3-(a-b)^2}{a+b}$.

9. Reduce $a+b-\dfrac{a^2-2ab+b^2}{a+b}$ to the form of a fraction.

Ans. $\dfrac{4ab}{a+b}$.

10. Reduce $x+2+\dfrac{x^2+4x+4}{x-2}$ to the form of a fraction.

Ans. $\dfrac{2x^2+4x}{x-2}$.

11. Reduce $a^2+2ab+b^2-\dfrac{a^3-3a^2b-3ab^2+b^3}{a+b}$ to the form of a fraction.

Ans. $\dfrac{6a^2b+6ab^2}{a+b}=6ab.$

12. Reduce $x+1+\dfrac{x+1}{x}$ to the form of a fraction.

Ans. $\dfrac{x^2+2x+1}{x}=\dfrac{(x+1)^2}{x}.$

CASE III.

(59.) *To reduce a fraction to an entire or mixed quantity.*

RULE.

Divide the numerator by the denominator, the quotient will be the entire quantity, and under the remainder, if any, write the denominator for the fractional part.

EXAMPLES.

1. Reduce $\dfrac{x^2+7x+14}{x+2}$ to a mixed quantity.

Operation.

$$\begin{array}{r|l} x^2+7x+14 & x+2=\text{divisor.} \\ \underline{x^2+2x} & \overline{x+5=\text{quotient.}} \\ 5x+14 & \\ \underline{5x+10} & \\ 4=\text{remainder.} & \end{array}$$

Therefore, the quantity required is, $x+5+\dfrac{4}{x+2}$

2. Reduce $\dfrac{a^2+2ab+b^2+2c}{a+b}$ to a mixed quantity.

Ans. $a+b+\dfrac{2c}{a+b}.$

3. Reduce $\dfrac{x^3+6x^2+12x+10}{x+2}$ to a mixed quantity.

Ans. $x^2+4x+4+\dfrac{2}{x+2}.$

4. Reduce $\frac{a^3+b^3}{a+b}$ to an entire quantity.

Ans. a^2-ab+b^2.

5. Reduce $\frac{a^5+b^5}{a+b}$ to an entire quantity.

Ans. $a^4-a^3b+a^2b^2-ab^3+b^4$.

6. Reduce $\frac{a^6-b^6}{a^2-b^2}$ to an entire quantity.

Ans. $a^4+a^2b^2+b^4$.

7. Reduce $\frac{x^3-y^3}{x+y}$ to a mixed quantity.

Ans. $x^2-xy+y^2-\frac{2y^3}{x+y}$.

8. Reduce $\frac{a^2+4ab+4b^2+c}{a+2b}$ to a mixed quantity.

Ans. $a+2b+\frac{c}{a+2b}$.

9. Reduce $\frac{a^3}{a+x}$ to a mixed quantity.

Ans. $a^2-ax+x^2-\frac{x^3}{a+x}$.

10. Reduce $\frac{x^2+12x+18}{x+3}$ to a mixed quantity.

Ans. $x+9-\frac{9}{x+3}$.

CASE IV.

(60.) To add fractions.

RULE I.

*Reduce the fractions to a common denominator, by multiplying the numerator and denominator of each fraction by the product of the denominators of the remaining fractions; then, add their numerators, and under their sum write the common denominator, which is the product of all the denominators.**

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* NOTE.—This rule may be explained by the 1st principle, Art. 56, for it is obvious that the terms of each fraction are multiplied by the same quantity. Rule II. is, in effect, the application of the same principle.

RULE II.

Find the least common multiple of the denominators, and it will be the least common denominator of the fractions. Divide the least common denominator by the denominator of each fraction, and multiply the quotients by the corresponding numerators of the given fractions, and the products will be the numerators, under each of which write the common denominator; then proceed as in Rule I.

EXAMPLES.

1. What is the sum of $\frac{a}{b}$, $\frac{m}{n}$, and $\frac{c}{x}$?

Here, $\frac{a}{b}=\frac{a\times(n\times x)}{b\times(n\times x)}=\frac{anx}{bnx}$; $\frac{m}{n}=\frac{m\times(b\times x)}{n\times(b\times x)}=\frac{bmx}{bnx}$; $\frac{c}{x}=\frac{c\times(b\times n)}{x\times(b\times n)}=\frac{bcn}{bnx}$. Hence, $\frac{a}{b}+\frac{m}{n}+\frac{c}{x}=\frac{anx}{bnx}+\frac{bmx}{bnx}+\frac{bcn}{bnx}=\frac{anx+bmx+bcn}{bnx}$, the sum required. Let the student observe, that by following the rule, the numerator and denominator of each of the given fractions are multiplied by the same quantity, and therefore the value of each fraction remains the same, as it should.

2. What is the sum of $\frac{a+b}{a-b}$, and $\frac{a-b}{a^3-b^3}$?

Operation.

$$a-b)\underline{a-b,\ a^3-b^3}$$
$$1,\ a^2+ab+b^2$$

$\therefore (a-b)\times(a^2+ab+b^2)=a^3-b^3=$the least common denominator.

$\frac{a^3-b^3}{a-b}\times\overline{a+b}=(a^2+ab+b^2)\times(a+b)=a^3+2a^2b+2ab^2+b^3$, the numerator of first fraction.

$\frac{a^3-b^3}{a^3-b^3}\times a-b=a-b$, the numerator of second fraction.

Whence, $\frac{a+b}{a-b}+\frac{a-b}{a^3-b^3}=\frac{a^3+2a^2b+2ab^2+b^3+a-b}{a^3-b^3}$, the fraction required.

3. What is the sum of $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$?

Ans. $\frac{2(a^2+b^2)}{a^2-b^2}$.

4. What is the sum of $\frac{a^2+2ab+b^2}{a-b}$ and $\frac{a^2-2ab+b^2}{a+b}$?

Ans. $\frac{2a^3+6ab^2}{a^2-b^2}$.

5. What is the sum of $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$?

Ans. $\frac{2(1+x^4)}{1-x^4}$.

6. What is the sum of $\frac{1}{1+x}$ and $\frac{1}{1-x}$? *Ans.* $\frac{2}{1-x^2}$.

7. What is the sum of $\frac{a}{bx}$, $\frac{d}{dx^2}$, and $\frac{c}{dx^3}$?

Ans. $\frac{adx^2+bdx+bc*}{bdx^3}$.

8. What is the sum of $\frac{1+x}{1-x}$, $\frac{1+x^2}{1-x^2}$, and $\frac{1+x^3}{1-x^3}$?

Ans. $\frac{3+5x+6x^2+5x^3+3x^4}{1+x-x^3-x^4}$.

9. What is the sum of $\frac{a^2+b^2}{a-b}$ and $\frac{a^2-b^2}{a+b}$?

Ans. $\frac{2(a^3+b^3)}{a^2-b^2}=\frac{2(a^2-ab+b^2)}{a-b}$.

10. What is the sum of $\frac{1}{x+y}$, $\frac{c}{x^3+y^3}$, and $\frac{1}{x+y}$?

Ans. $\frac{2(x^2-xy+y^2)+c}{x^3+y^3}$.

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* Note.—If Rule I. be used the fraction obtained by it must be reduced to its lowest terms, in order to agree with the given answer.

11. What is the sum of $\frac{3}{48a^2}$, $\frac{4}{64ax}$, and $\frac{5}{12a^2x}$?

Ans. $\frac{12x^2+12ax+80x}{192a^2x^2}$.

12. What is the sum of $\frac{a}{6x^2}$, $\frac{c}{7ax}$, and $\frac{n}{21ax^2}$?

Ans. $\frac{7a^2+6cx+2n}{42ax^2}$.

13. What is the sum of $\frac{3x}{5}$, $\frac{4x}{12}$, and $\frac{7x}{60}$?

Ans. $\frac{36x+20x+7x}{60}=x+\frac{x}{20}$.

14. What is the sum of $\frac{1}{a^2-b^2}$ and $\frac{1}{(a-b)^2}$?

Ans. $\frac{2a}{a^3-a^2b-ab^2+b^3}$.

CASE V.

(61.) To subtract one fraction from another.

RULE.

Reduce the fractions to a common denominator, if they have not one, and then subtract the numerator of the subtrahend from that of the minuend, and place the difference over the common denominator.

EXAMPLES.

1. From $\frac{2x}{c}$ take $\frac{3x}{c^3}$.

Here $\frac{2x}{c}=\frac{2x\times c^2}{c\times c^2}=\frac{2c^2x}{c^3}$. Hence, $\frac{2x}{c}-\frac{3x}{c^3}=\frac{2c^2x}{c^3}-\frac{3x}{c^3}=\frac{2c^2x-3x}{c^3}$, the difference required.

2. From $\frac{a+b}{2}$ take $\frac{a-b}{2}$. *Ans.*

3. From $\frac{10x}{9}$ take $\frac{2x}{3}$. *Ans.* $\frac{4x}{9}$.

4. From $\frac{7a}{3y}$ take $\frac{2a}{4y}$. *Ans.* $\frac{11a}{6y}$.

5. From $\frac{5x+3}{24}$ take $\frac{x-3}{16}$. *Ans.* $\frac{7x+15}{48}$.

6. From $\frac{3x+2}{a}$ take $\frac{7ax-10a}{a^2}$.

Ans. $\frac{12a-4ax}{a^2}=\frac{4(3-x)}{a}$.

7. From $\frac{2x+1}{12}$ take $\frac{3x-2}{18}$. *Ans.* $\frac{7}{36}$.

8. From $\frac{a+b}{a-b}$ take $\frac{a-b}{a+b}$. *Ans.* $\frac{4ab}{a^2-b^2}$.

9. From $\frac{a+b}{a^2-b^2}$ take $\frac{a+b}{a^2-2ab+b^2}$.

Ans. $\frac{-2b(a+b)}{a^3-a^2b-ab^2+b^3}$.

10. From $\frac{a^4+b^4}{a-b}$ take $\frac{a^4-b^4}{a+b}$. *Ans.* $\frac{2ab(a^3+b^3)}{a^2-b^2}$.

11. From $\frac{1}{a-b}$ take $\frac{1}{a^2-2ab+b^2}$.

Ans. $\frac{a-b-1}{a^2-2ab+b^2}$.

12. From $\frac{1}{x+4}$ take $\frac{-x^2-3x+2}{x^2+10x+24}$.

Ans. $\frac{(x+2)^2}{x^2+10x+24}$.

13. From $\frac{1+x^2}{1-x^2}$ take $\frac{1-x^2}{1+x^2}$. *Ans.* $\frac{4x^2}{1-x^4}$.

14. From $\frac{2x+5}{24}$ take $\frac{3x+2}{72}$. *Ans.* $\frac{3x+13}{72}$.

15. From $\frac{1}{a^2-b^2}$ take $\frac{1}{a^2+2ab+b^2}$.

Ans. $\frac{2b}{a^3+a^2b-ab^2-b^3}$.

CASE VI.

(62.) To multiply one fraction by another.

RULE.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.

Let it be required to find the product of the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

In the first place, multiply the fraction $\frac{a}{b}$ by c, the numerator of the second fraction. By the second principle, Art. 56, this product is, $\frac{a\times c}{b}=\frac{ac}{b}$. Now, since it was required to multiply $\frac{a}{b}$ by the dth part of c, it follows that the product, $\frac{ac}{b}$ must be d times too large. Therefore, in order to obtain the true product, $\frac{ac}{b}$ must be divided by d. By the third principle, Art. 56, $\frac{ac}{b}\div d=\frac{ac}{b\times d}=\frac{ac}{bd}$. Hence the rule is correct.

(63.) By the first principle, Art. 56, we may, in the multiplication of fractions, cancel all the factors which are common to both numerators and denominators.

EXAMPLES.

1. Multiply $\frac{a^2-x^2}{a+c}$ by $\frac{3a^2+3ac}{4ax+4a^2}$.

By resolving a^2-x^2, $3a^2+3ac$, and $4ax+4a^2$ into factors, and then multiplying, we have

$$\frac{(a+x)(a-x)}{a+c}\times\frac{3a(a+c)}{4a(a+x)}=\tfrac{3}{4}(a-x)$$

2. Multiply $\frac{48a^2}{32n^3}$ by $\frac{8n^2}{64a^3}$. *Ans.* $\frac{3}{16an}$.

3. Multiply $\frac{7x}{21}$ by $\frac{13}{18x^2}$. *Ans.* $\frac{13}{54x}$.

4. Multiply $\frac{42m^2}{64n^2}$ by $\frac{80a^2}{21b^2}$. *Ans.* $\frac{5a^2m^2}{2b^2n^2}$.

5. Multiply $\frac{am+m^2}{ax^2}$ by $\frac{x}{mc}$. *Ans.* $\frac{a+m}{acx}$.

6. Multiply $\frac{ny^2+y^3}{acx^4}$ by $\frac{x^2}{3y^2}$. *Ans.* $\frac{n+y}{3acx^2}$.

7. Multiply $\frac{17a}{20b}$ by $\frac{5c}{34a^2}$. *Ans.* $\frac{c}{8ab}$.

8. Multiply $a+\frac{a}{ax}$ by $c+\frac{c}{ay}$.*

Ans. $\frac{a^2cyx+acx+acy+c}{ayx}$.

9. Multiply $\frac{a+b}{a^3+b^3}$ by $\frac{2a^2-2ab+2b^2}{5}$. *Ans.* $\frac{2}{5}$.

10. Multiply $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$. *Ans.* $\frac{a^2+b^2}{a^2+2ab+b^2}$.

11. Multiply $\frac{x^2-9x+20}{x^2-6x}$ by $\frac{x^2-13x+42}{x^2-5x}$.

Ans. $\frac{x^2-11x+28}{x^2}$.

12. Multiply $\frac{x^2+3x+2}{x^2+2x+1}$ by $\frac{x^2+5x+4}{x^2+7x+12}$.

Ans. $\frac{x+2}{x+3}$.

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* NOTE.—Before multiplying, reduce the mixed quantities to improper fractions.

13. Multiply $\dfrac{a^2-b^2}{b^2}$ by $\dfrac{a}{a+b}+\dfrac{b^2-ab}{a^2-b^2}$.

Ans. $\dfrac{a^2-2ab+b^2}{b^2}$.

14. Multiply $\dfrac{a^3+b^3}{a+b}$ by $\dfrac{(a+b)^2}{a^2-ab+b^2}$. *Ans.* $(a+b)^2$.

15. Multiply $\dfrac{5y}{x^2+4x+4}$ by $\dfrac{5ax+10a}{7m}$.

Ans. $\dfrac{25ay}{7mx+14m}$.

CASE VII.

(64.) To divide one fraction by another.

RULE.

Invert the divisor, and then proceed as in multiplication.

Let it be required to find the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$. In the first place, divide $\frac{a}{b}$ by c, the numerator of the divisor. By the third principle, Art. 56, $\frac{a}{b}\div c=\frac{a}{b\times c}=\frac{a}{bc}$. Since the divisor, c, is d times as large as the true divisor, the quotient, $\frac{a}{bc}$, must be one dth part of the true quotient. Hence, the true quotient is, $\frac{a}{bc}\times d=\frac{ad}{bc}$. This is the same result as may be obtained by the rule, and therefore the rule is correct.

EXAMPLES.

1. Divide $\dfrac{a+b}{a-b}$ by $\dfrac{a+b}{a-b}$. *Ans.* 1.

2. Divide $\dfrac{a-b}{a+b}$ by $\dfrac{a^2+2ab+b^2}{a+b}$.

Ans. $\dfrac{a-b}{(a+b)^2}$.

3. Divide $\frac{5x+4}{48}$ by $\frac{3x+2}{16}$. *Ans.* $\frac{5x+4}{9x+6}$.

4. Divide $\frac{5x+10}{a+b}$ by $\frac{x+2}{x^2}$. *Ans.* $\frac{5x^2}{a+b}$.

5. Divide $\frac{14x+28}{96}$ by $\frac{42x+84}{72}$. *Ans.* $\frac{1}{4}$.

6. Divide $\frac{a^3+a^2x}{2c^2}$ by $\frac{a^2}{4c^2+c}$. *Ans.* $\frac{4ac+4cx+a+x}{2c}$.

7. Divide $\frac{a^2+2ab+b^2}{a-b}$ by $\frac{a^2-b^2}{4}$. *Ans.* $\frac{4a+4b}{a^2-2ab+b^2}$.

8. Divide $\frac{a^4-b^4}{a^2-2ab+b^2}$ by $\frac{a^2+ab}{a-b}$.

Ans. $\frac{a^2+b^2}{a}=a+\frac{b^2}{a}$.

9. Divide $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a+b}{a-b}$. *Ans.* $\frac{a^2+b^2}{a^2+2ab+b^2}$.

10. Divide $\frac{x^2-9x+20}{x^2-6x}$ by $\frac{x^2-5x}{x^2-13x+42}$.

Ans. $\frac{x^2-11x+28}{x^2}$.

11. Divide $1+\frac{n-1}{n+1}$ by $1-\frac{n-1}{n+1}$. *Ans.* n.

12. Divide $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a-b}-\frac{b}{a+b}$. *Ans.* 1.

13. Divide $\frac{a^2-b^2}{a^3+b^3}$ by $\frac{a^3-b^3}{a^6-b^6}$. *Ans.* a^2-b^2.

14. Divide $\frac{a+b}{a-b}$ by $\frac{a-b}{a+b}$. *Ans.* $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

15. Divide $\frac{ac+c^2}{24}$ by $\frac{4c}{12a+12c}$. *Ans.* $\frac{(a+c)^2}{8}$.

16. Divide $\frac{a+x}{a-x}+\frac{a-x}{a+x}$ by $\frac{a+x}{a-x}-\frac{a-x}{a+x}$. *Ans.* $\frac{a^2+x^2}{2ax}$.

CHAPTER IV.

SIMPLE EQUATIONS.

(65.) An equation is an algebraic expression, consisting of two equal quantities, with the sign of equality placed between them. Thus, $5x+3=14$ is an equation.

(66.) The two quantities which are separated by the sign of equality are called the members of the equation. That on the left of the sign of equality is called the *first member*, and that on the right, the *second member*.

(67.) When one member of an equation is a repetition of, or the result of some operation *indicated* in, the other member, it is called an *identical* equation. Thus, $(x^2-y^2)\div(x-y)=x+y$ and $5x+4=5x+4$ are identical equations.

(68.) Equations are divided into degrees. Those which contain only the first power of the unknown quantity are simple equations, or equations of the first degree. Those which contain the square of the unknown quantity, and no higher power, are quadratic equations, or equations of the second degree; and, in general, those equations which contain the nth power of the unknown quantity, and no higher power, are equations of the nth degree. Thus, $5x+2x+5=15$ is an equation of the first degree, $3x^2+4x=20$ is an equation of the second degree, and $x^n+34=a$ is an equation of the nth degree.

(69.) Any value of the unknown quantity, which, when it is substituted for the unknown quantity, will satisfy the equation, that is, render the two members identical, is a root of that equation. Thus, 3 is the root of the equation, $2x+4=10$, since, if in the place of $2x$, 2×3, or 6, be substituted, the equation be-

comes $6+4=10$, an identical equation. The solution of an equation consists in finding all of its roots. A simple equation has only one root.

SOLUTION OF SIMPLE EQUATIONS.

(**70.**) The rules for solving equations are founded upon the following axioms:

1. *If the same quantity be added to equals, the sums will be equal.*

2. *If the same quantity be subtracted from equals, the remainders will be equal.*

3. *If equal quantities be multiplied by the same quantity, the products will be equal.*

4. *If equal quantities be divided by the same quantities, the quotients will be equal.*

TRANSPOSITION.

(**71.**) By the term *transposition* is meant, the changing of terms in one member of an equation into the other member, without destroying the equality.

Let it be required to find the value of x in the equation, $x+8=12$. Subtract 8 from each member, and we have $x+8-8=12-8$, or $x=4$, the value required. This result may be obtained by transposing 8 into the second member, and changing its sign. If the equation had been $x-8=12$, it would have been necessary to add 8 to each member in order to have obtained the value of x, or the result might have been obtained by transposing 8 into the other member and changing its sign. Hence, we may observe that

A quantity may be transposed from one member to the other by changing its sign.

TO CLEAR AN EQUATION OF FRACTIONS.

(**72.**) Let it be required to clear the equation, $\frac{x}{4}+\frac{x}{3}+\frac{x}{24}=15$, of fractions. This object may obviously be accomplished by

multiplying each member by any multiple of the denominators. Therefore, multiply each member by 24, the least common multiple of the denominators, and it becomes, $6x+8x+x=360$. As any equation may be cleared of fractions in a similar manner, we have the following

RULE.

Multiply each member of the equation by any multiple of the denominators, or by the least common multiple of all the denominators.

(73.) *To find the value of the unknown quantity.*

Let it be required to find the value of x in the equation,

$$\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7 \qquad (1)$$

$$6x+4x=3x+84 \qquad (2)=(1)\times 12,$$ that is, the second equation is equal to the first, multiplied by 12, the least common multiple of the denominators.

$$6x+4x-3x=84 \qquad (3)$$

The third equation is obtained by transposing $3x$, which is found in the second member of equation (2), into the first member, and changing its sign.

$$7x=84 \qquad (4)$$

The fourth equation is obtained from the third, by uniting the terms in the first member of the third equation. Thus, $6x$ and $4x$ are $10x$, and $3x$ taken from $10x$ leaves $7x$.

$$x=12 \qquad (5)=(4)\div 7$$

That is, the fifth equation is obtained by dividing each member of the fourth by 7.

From what has been said, we derive the following general

RULE.

1. *Clear the equation of fractions, if it have any, and perform all the operations indicated.*

2. *Transpose all the unknown quantities into the first member, and all the known quantities into the second member.*

3. *Unite the terms in each member, and divide by the coefficient of the unknown quantity.*

EXAMPLES.

1. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find the value of x.

Multiply each member by 16, the least common multiple of the denominators, and we have

$$336+3x-11=10x-10+776-56x \quad (2)$$

By transposing in (2), $3x-10x+56x=-10+776-336+11$ (3)

Uniting terms in (3), $49x=441$ (4)

Whence, by dividing by 49, $x=9$ (5)

2. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$ to find x.

Multiply each member by 28, and the equation becomes

$$18x-19+\frac{154x+294}{3x+7}=18x+30$$

By transposing $\frac{154x+294}{3x+7}=49$

Dividing by 7 $\frac{22x+42}{3x+7}=7$

Clearing of fractions, $22x+42=21x+49$

By transposing, $x=7$

It may be observed that the solution of this example is abbreviated by partially clearing, at first, the equation of fractions.

3. Given $\frac{x}{3}+\frac{x}{8}+\frac{x}{9}+\frac{x}{12}+\frac{x}{18}=51$, to find x.

Multiply by 72, $24x+9x+8x+6x+4x=51\times72$
Uniting terms, $51x=3672$
Dividing by 51, $x=72$

4. Given $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$ to find x.

Multiply each number by 9, and we have

$$6x+7+\frac{63x-117}{6x+3}=6x+12$$

Subtract $6x+7$ from each member, $\frac{63x-117}{6x+3}=5$
Multiply by $6x+3$, $63x-117=30x+15$
By transposing, $63x-30x=132$
Or, $33x=132$
Whence, $x=4.$*

5. Given $(a+x)(a-x)+2abx=2a^2+b^2-x^2$ to find x.

Performing the multiplication indicated, we have

$$a^2-x^2+2abx=2a^2+b^2-x^2$$

By transposing, $2abx=a^2+b^2$
Whence, $x=\frac{a^2+b^2}{2ab}$

6. Given $a+x=\frac{x^2+2ab}{a+x}$ to find x.

Multiply by $a+x$, $a^2+2ax+x^2=x^2+2ab.$
By transposing, $2ax=2ab-a^2$
Whence, $x=\frac{2ab-a^2}{2a}=\frac{2b-a}{2}.$

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* NOTE.—In the second and fourth examples, we have employed some artifices, in order to abbreviate the solutions. The student must learn to rely on his own ingenuity in solving examples and problems, as no general rule can be given.

7. Given $\frac{7-x}{2}+4=\frac{6x-22}{8}+\frac{8x+15}{6}$ to find x.

Ans. $x=3$.

8. Given $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$ to find x.

Ans. $x=12$.

9. Given $x-\frac{2x+1}{3}=\frac{x+3}{4}$ to find x. *Ans.* $x=13$.

10. Given $\frac{3x-3}{4}-\frac{3x-3}{3}=\frac{15}{3}-\frac{27+4x}{9}$ to find x.

Ans. $x=9$.

11. Given $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$ to find x.

Ans. $x=4$.

12. Given $\frac{20x+36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$ to find x.

Ans. $x=4$.

13. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$ to find x.

Ans. $x=8$.

14. Given $\frac{x-1}{7}+\frac{23-x}{5}=7-\frac{4+x}{4}$ to find x.

Ans. $x=8$.

15. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find x.

Ans. $x=9$.

16. Given $23+\frac{5x-1}{11}+\frac{3x-2}{5}-\frac{11x-3}{12}=\frac{13x-15}{3}-\frac{8x-2}{7}$ to find x.

Ans. $x=9$.

17. Given $4x+\frac{1}{10}-\frac{3x-12}{16}-\frac{12+7x}{9}=7x-33-\frac{9+5x}{10}-\frac{11x-17}{8}$ to find x.

Ans. $x=15$.

18. Given $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}$ to find x.

Ans. $x=3$.

19. Given $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}$ to find x. *Ans.* $x=\frac{b}{c}$.

20. Given $\frac{5x-25}{20}+6x=\frac{284-x}{5}$ to find x.

Ans. $x=9$.

21. Given $\frac{x}{7}+\frac{x}{3}+\frac{x}{9}+\frac{x}{21}=40$ to find x. *Ans.* $x=63$.

22. Given $\frac{x}{3}+\frac{x}{5}+\frac{x}{7}+\frac{x}{11}=886$ to find x.

Ans. $x=1155$.

23. Given $\frac{x}{6}+\frac{x}{7}+\frac{x}{4}+\frac{x}{12}+\frac{x}{42}+\frac{x}{3}=84$ to find x.

Ans. $x=84$.

24. Given $\frac{a^2+2ax+x^2}{4}=abx+\frac{3x^2}{12}$ to find x.

Ans. $x=\frac{a}{4b-2}$.

25. Given $x^2\sqrt{3}=ax+bx+cx$ to find x.

Ans. $x=\frac{a+b+c}{\sqrt{3}}$.

26. Given $ax=ab-bx$ to find x. *Ans.* $x=\frac{ab}{a+b}$.

27. Given $\frac{3x+4}{5}-\frac{7x-3}{2}=\frac{x-16}{4}$ to find x.

Ans. $x=2$.

28. Given $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$ to find x.

Ans. $x=4$.

29. Given $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x$ to find x.

Ans. $x=4$.

30. Given $\frac{7x+5}{3}-\frac{16+4x}{5}+6=\frac{3x+9}{2}$ to find x.

Ans. $x=1$.

31. Given $\frac{5x+3}{4}+7=\frac{4x-10}{10}+10$ to find x.

Ans. $x=\frac{25}{17}$.

32. Given $\frac{x}{6}-\frac{x}{4}=\frac{x}{3}-\frac{x}{2}+1$ to find x. *Ans.* $x=12$.

SOLUTION OF QUESTIONS WHICH INVOLVE ONE UNKNOWN QUANTITY.

(74.) The solution of a problem by algebra consists in expressing the relations which exist between the known and unknown quantities by means of equations, and then obtaining the value of the unknown quantities from these equations.

To obtain the equations is generally the most difficult part of the labor, but sometimes the solution of them presents the greater difficulty. In many problems, the enunciation readily furnishes the necessary equations, and then the conditions are said to be *explicit*. *Implicit* conditions are those which are deduced from explicit conditions.

No general rule can be given for solving problems. We shall give a few for the purpose of initiating the student, and then he must depend on his own ingenuity and powers of analysis. He will find that much practice and reflection are necessary in order to become a good resolver of problems.

PROBLEMS.

1. What two numbers are those whose difference is 7, and sum 33?

Let x = the less number;
Then $x+7$ = the greater number.

$x+x+7=33$, by the conditions of the problem.

By uniting terms and transposing,

$$2x=26;$$

Whence, $\left.\begin{array}{l} x=13 \\ x+7=20 \end{array}\right\}$ Ans.
and

2. Out of a cask of wine which had leaked away $\frac{1}{3}$, 21 gallons were drawn, and then the cask was found to be half full; how much did it hold?

Let x = number of gallons which it contained;

Then $\frac{x}{3}$ = number of gallons which leaked away;

Whence, by the conditions of the question,

$$\frac{x}{2}=\frac{x}{3}+21$$

Multiplying by 6, we have

$$3x=2x+126$$

By transposing, $x=126$ Ans.

3. A gentleman has stocks upon which he receives an annual dividend of 12 per cent. He spends $\frac{1}{8}$ of his yearly income for clothes, and $\frac{1}{2}$ of it for the support of his family, and the remainder, which is \$360, he gives to the poor. What is the amount of his property?

Let x = the amount of his property;

Then $x\times\frac{12}{100}=\frac{3x}{25}$ = his yearly income,

$\frac{3x}{25}\times\frac{1}{8}=\frac{3x}{200}$ = the cost of his clothing,

and $\frac{3x}{25}\times\frac{1}{2}=\frac{3x}{50}$ = his family expenses.

Whence, by conditions of the question,

$$\frac{3x}{200}+\frac{3x}{50}+360=\frac{3x}{25} \quad (1)$$

Or, $3x+12x+360\times200=24x$ (2)

By transposing, $9x=360\times200$ (3)

Whence, $x=40\times200=8000$ Ans.

4. A merchant supports himself for three years for a dollars a year, and at the end of each year increases his stock, which was not thus expended by one third part of the remainder. At the end of the third year his original stock was doubled. What was his stock?

Let x=his stock;

Then, $(x-a)\times\frac{4}{3}=\frac{4x-4a}{3}$=his stock at the end of 1st year;*

$\left(\frac{4x-4a}{3}-a\right)\times\frac{4}{3}=\frac{16x-16a}{9}-\frac{4a}{3}=$ " " 2d year;

$\left(\frac{16x-16a}{9}-\frac{4a}{3}-a\right)\times\frac{4}{3}=\frac{64x-64a}{27}-\frac{16a}{9}-\frac{4a}{3}=$ " 3d year;

Whence, by conditions of the question,

$$\frac{64x-64a}{27}-\frac{16a}{9}-\frac{4a}{3}=2x \quad (1)$$

$$64x-64a-48a-36a=54x \quad (2)$$

By transposing and uniting terms in (2), we have

$$10x=148a \quad (3)$$

Whence, $x=\frac{148a}{10}=\frac{74a}{5}$ Ans.

5. A hare pursued by a greyhound is 60 of her own leaps in advance of the dog; she makes 9 leaps during the time that the greyhound makes 6; but 3 leaps of the greyhound are equivalent to 7 of the hare. How many leaps must the greyhound take before he overtakes the hare?

Let $6x$=the number of leaps that the dog takes,

Then $9x$= " " " hare takes after the dog starts.

7 of the hare's leaps=3 of the dog's leaps;

$\therefore$ 1 " " $=\frac{3}{7}$ of one of the dog's leaps;

Whence $9x+60$ " $=\frac{3}{7}\times(9x+60)$ of the dog's leaps.

Therefore, by conditions of the question,

$$6x=\tfrac{3}{7}(9x+60) \quad (1)$$

Or, $$42x=27x+180 \quad (2)$$

By transposing, $$15x=180 \quad (3)$$

$$\therefore\ x=12 \quad (4)$$

$$6x=72 \text{ Ans.} \quad (5)$$

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* NOTE.—A quantity may be increased by one third part of itself by multiplying it by $\frac{4}{3}$.

In this example, it is necessary to express the leaps of the hare in terms of the leaps of the dog, in order that we may commit no absurdity by equating heterogeneous numbers, or numbers which are not related to the same unit of measure.

6. A can perform a piece of work in 8 days, and B can perform the same work in 24 days. In what time will they finish it if both work together?

Let x=the time required.

Since A can perform the work in 8 days, in one day he can perform $\frac{1}{8}$ of it, and in x days he can perform x times $\frac{1}{8}$ of the work, or $\frac{x}{8}$ of it. In the same way we find that $\frac{x}{24}$ is the part of the work that B can do in x days. Now, the part that A can do, added to the part that B can do, must equal the whole work, or unity.

Whence $\qquad \frac{x}{8}+\frac{x}{24}=1 \qquad (1)$

Clearing of fractions, $\qquad 3x+x=24 \qquad (2)$

Or, $\qquad 4x=24 \qquad (3)$

$$x=6 \text{ Ans.} \qquad (4)$$

7. A laborer engaged to work for n days. For each day that he labored he received a cents, and for each day that he was idle he forfeited b cents. Now, at the end of the time he received c cents. It is required to find how many days he worked, and how many he was idle.

Let x=the number of days which he worked;
Then $n-x$= " " " was idle.
ax=what he earned.
$b\times(n-x)$=what he forfeited.

Whence, by conditions of the question,

$$ax-b(n-x)=c \qquad (1)$$

Or, $\qquad ax-bn+bx=c \qquad (2)$

Transposing, $\qquad ax+bx=c+bn \qquad (3)$

$$x=\frac{c+bn}{a+b}, \text{ the number of working days.}$$

$$\therefore\ n-x=n-\frac{c+bn}{a+b}=\frac{an-c}{a+b}, \text{ the number of idle days.}$$

$$\left.\begin{aligned}\text{If we make } n&=40\\ a&=50 \text{ cents}\\ b&=50 \text{ cents}\\ c&=400 \text{ cents}\end{aligned}\right\}\ \begin{aligned}&\text{We find that } x=24 \text{ days,}\\ &\text{and }\ n-x=16 \text{ days.}\end{aligned}$$

8. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 11 feet out of water; what is the whole length? *Ans.* 20 feet.

9. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse; what was in it at first? *Ans.* 120 guineas.

10. Two persons, A and B, lay out equal sums of money in trade; A gains \$126, and B loses \$87, and A's money is now double that of B's; what did each lay out? *Ans.* \$300.

11. Divide the number 54 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may all equal each other. *Ans.* 12, 18, 24.

12. A person was desirous of giving 3 cents apiece to some beggars, but found that he had not money enough in his pocket by 8 cents, he therefore gave each one 2 cents, and he then had 3 cents remaining; required the number of beggars. *Ans.* 11.

13. A person in play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had, and then won 2 shillings; lastly, he lost $\frac{1}{7}$ of what he then had, and then he found that he had 12 shillings remaining; what had he at first? *Ans.* 16 shillings.

14. Divide the number a into two parts which shall have to each other the ratio of m to n. *Ans.* $\frac{ma}{m+n}, \frac{na}{m+n}$.

15. Divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, shall all be equal to each other. What are the parts? *Ans.* 18, 22, 10, 40.

16. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days. In how many days would the man alone empty the cask? *Ans.* 20 days.

17. The hour and minute hand of a clock are exactly together at 12 o'clock; when are they next together?

Ans. 1 hour and $5\frac{5}{11}$m.

18. Divide the number a into three parts which shall be to each other as m, n, p.

$$Ans.\ \frac{ma}{m+n+p},\ \frac{na}{m+n+p},\ \frac{pa}{m+n+p}.$$

19. Two travellers set out at the same time from London and York, whose distance is 150 miles; one of them goes 8 miles a day, and the other 7; when will they meet?

Ans. In 10 days.

20. The sum of $1320 is to be divided between A and B in such proportion, that as often as A receives an eagle B receives a dollar. What did each receive?

Ans. A $1200, B $120.

21. Four places are situated in the order of the letters A, B, C, D. The distance from A to D is 34 miles, the distance from A to B is to the distance from C to D as 2 is to 3, and one fourth of the distance from A to B, added to one half of the distance from C to D, is three times the distance from B to C. What are the respective distances?

Ans. A B$=$12, B C$=$4, and C D$=$18 miles.

22. After paying $\frac{1}{9}$ of my money, and then $\frac{1}{8}$ of the remainder, I had $140 left. How much money had I at first?

Ans. $180.

23. Two numbers are to each other as a to b, and if c be added to each, the first sum will be n times the second sum. What are the numbers? $Ans.\ \frac{ac(n-1)}{a-nb}$ and $\frac{bc(n-1)}{a-nb}$.

24. A cistern has two pipes; the first will fill it in 40 minutes, and the second in 60 minutes. In what time can the cistern be filled if both pipes are open? *Ans.* 24 minutes.

25. A cistern has three pipes; the first will fill it in a minutes, the second in b minutes, and the third in c minutes. In what time can the cistern be filled if the three pipes are open at the same time? *Ans.* $\frac{abc}{ab+ac+bc}$.

26. Twelve oxen can eat up $3\frac{1}{3}$ acres of grass in 4 weeks, together with that which grows during the time that they are grazing; in the same manner, 21 oxen can eat 10 acres in 9 weeks. How many oxen can, in this way, eat up 24 acres in 18 weeks? *Ans.* 36 oxen.

27. A cask that held 146 gallons, was filled with a mixture of brandy, wine, and water; there were 15 gallons of wine more than there were of brandy, and the number of gallons of water was equal to the whole number of gallons of wine and brandy. What was the number of gallons of each?

Ans. 29 gallons brandy, 44 wine, 73 water.

28. The rent of an estate is 8 per cent. greater this year than it was last, and this year it is $1890. What was it last year?

Ans. $1750.

29. A's age is double of B's, B's is triple of C's, and the sum of all their ages is 140 years. What is the age of each?

Ans. A's$=84$, B's$=42$, and C's$=14$.

30. There is a fish whose tail weighs 9 lbs., his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail. What is the whole weight of the fish? *Ans.* 72 lbs.

31. A field of 864 square rods is to be divided among three farmers, A, B, and C, so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How much does each receive?

Ans. A 135, B 297, C 432 square rods.

32. On an approaching war, three towns, A, B, C, are to furnish 594 soldiers; the division is to be made in proportion to their population. Now, the population of A is to that of B as 3 is to 5, and the population of B is to that of C as 8 to 7. How many men must each town furnish?

Ans. A 144, B 240, C 210 men.

33. Divide the number a into three such parts that the first may be to the second as m to n, and the second part to the third as p to q.

Ans. $\dfrac{mpa}{mp+np+nq}, \dfrac{npa}{mp+np+nq}, \dfrac{nqa}{mp+np+nq}$.

34. A capital was put out for one year at $4\frac{1}{2}$ per cent. per annum; at the end of the year there was received $13167, in capital and interest. How much did the capital amount to?

Ans. 12600.

35. A father leaves a number of children, and a certain sum, which they are to divide amongst them as follows: the first is to receive $100, and the 10th part of the remainder, after this, the second is to receive $200, and the 10th part of what then remains, again, the third receives $300, and the 10th part of the remainder; and so on, each succeeding child receives $100 more than the one immediately preceding, and then the 10th part of the remainder. At last it is found that each child receives the same. What was the fortune left, and how many children were there?

Ans. The fortune was $8100, and the number of children was 9.

36. Further, what must the fortune and the number of children be, when, in general, the first receives a dollars, together with the nth part of the remainder; and each succeeding child a dollars more, together with the nth part of the remainder, and it is found at last that each has received the same?

Ans. The fortune$=(n-1)^2a$; the number of children$=n-1$.

37. Find two such numbers that the one may be m times as great as the other, and their sum$=a$. What are these two numbers?

Ans. $\dfrac{a}{m+1}$ and $\dfrac{ma}{m+1}$.

38. What capital is that which being put at interest for 5 years at 4 per cent., amounts to \$8208? *Ans.* \$6840.

39. A person puts a capital of \$5500 out at interest at 4 per cent., and $4\frac{1}{2}$ years after, another capital of \$8000 at 5 per cent. If he leaves these two capitals constantly at interest, in how many years will he have drawn the same interest from both?

Ans. In 10 years from the time he put out the first sum.

40. A merchant adds yearly to his capital one third, but takes from it, at the end of each year, \$1000 for his expenses. At the end of the third year he finds that his original stock was doubled. What was his original capital? *Ans.* \$11100.

41. A general, wishing to draw up his regiment in the form of a square, tried it in two ways; the first time he had 39 men over, the second time, having extended the side of the square by one man, he wanted 50 men to complete the square. What was the number of men in the regiment? *Ans.* 1975 men.

42. It is required to find a number, such, that if the two numbers, a and b, are added to it, the difference between the squares of these sums may be$=d$. What is the number?

Ans. $\dfrac{d-a^2+b^2}{2(a-b)}$.

Does the solution of this problem include that of the preceding one?

43. Three merchants, A, B, C, enter into partnership. A advances \$1200, B \$800, C \$600. A leaves his money in trade 8 months, B 10 months, and C 14 months. They gain \$500. What is each one's share of the gain?

Ans. A $\$184\frac{8}{13}$, B $\$153\frac{11}{13}$, C $161\frac{7}{13}$.

44. Three merchants entered into trade; the first contributed \$17000, the second \$13000, and the third \$10000. As they must have some person to conduct the business, the one who had furnished the least offered to undertake the management of it, on the condition that he shall receive 3 per cent. profit, besides what he is entitled to from his deposited capital. Now, it is found that they have gained \$35262.50; how much is due to each?

Ans. To the 1st \$14875, to the 2d \$11375, to the 3d \$9012.50.

45. A person possesses a wagon with a mechanical contrivance by which the difference in the number of revolutions made by the fore and hind wheels may be determined. The fore wheel is a feet, and the hind wheel is b feet in circumference. What is the distance gone over, when the fore wheel has made n revolutions more than the hind wheel?

Ans. $\frac{abn}{b-a}$ feet.

46. Two bombardiers threw different kinds of shells from a battery; the first had thrown 36 times before the second commenced, and he throws 8 times while the second throws 7 times; but the second uses as much powder for three of his throws as the first does for four. How many throws must the second make in order to consume as much powder as the first?

Ans. 189 throws.

47. Saltpetre and sulphur are mixed together in a mass of 80 lbs., and in such a proportion that for every 7 parts of saltpetre there are three parts of sulphur. How much saltpetre must be added to the mass, so that the proportion of these substances may be such that for every 11 parts of saltpetre there may be 4 parts of sulphur?

Ans. 10 lbs.

48. A courier who started from a certain place 10 days ago, is pursued by another from the same place, and by the same way. The first goes 4 miles every day, and the second 9. How many days will the second need to overtake the first?

Ans. 8 days.

49. A courier left this place n days ago, and goes a miles each day. He is pursued by another going b miles daily. How many days will the second require to overtake the first?

Ans. $\frac{na}{b-a}$ days.

50. In a full wine cask there are three faucets; by the first the wine can be drawn off in 2, by the second in 3, and by the third in 4 hours. What time will be required to empty the cask when all three are running at once? *Ans.* $55\frac{5}{13}$ minutes.

51. At 12 o'clock both hands of a clock are together. When, and how often will these hands be together during the next 12 hours?

Ans. The hands will be together at $5\frac{5}{11}$ minutes past 1, $10\frac{10}{11}$ minutes past 2, $16\frac{4}{11}$ minutes past 3, and so on, in each successive hour $5\frac{5}{11}$ minutes later; and the hands will be together 11 times.

52. Two bodies move after one another in the circumference of a circle which measures a feet. At first they are distant from each other by an arc which measures b feet; the first moves c feet, and the second moves C feet in a second. When will these two bodies meet for the first time, second time, and so on, supposing that they do not disturb each other's motion?

Ans. In $\frac{b}{C-c}, \frac{b+a}{C-c}, \frac{2a+b}{C-c}$, &c.

53. When will they meet if the first starts t seconds later than the second? *Ans.* $\frac{b-ct}{C-c}, \frac{a+b-ct}{C-c}, \frac{2a+b-ct}{C-c}$, &c.

54. When will they meet if the first begins to move t seconds sooner than the second?

Ans. $\frac{b+ct}{C-c}, \frac{a+b+ct}{C-c}, \frac{2a+b+ct}{C-c}$, &c.

55. When will they meet, if the first, instead of preceding the second, runs against it, and starts from the same place t seconds sooner? *Ans.* $\frac{b-ct}{C+c}, \frac{a+b-ct}{C+c}, \frac{2a+b-ct}{C+c}$.

56. When will they meet if the first starts from the same place as the second, and begins to move t seconds later?

Ans. $\dfrac{b+ct}{C+c}, \dfrac{a+b+ct}{C+c}, \dfrac{2a+b+ct}{C+c}$, &c.

57. A wine merchant has two kinds of wine; the one cost 9 shillings per gallon, the other 5. He wishes to mix both sorts together, in such quantities, that he may have 50 gallons, and each gallon, without profit or loss, may be sold for 8 shillings. How much must he take of each sort to make up this mixture?

Ans. $37\frac{1}{2}$ gallons of the best, $12\frac{1}{2}$ of the other.

58. Let the price of the best wine in the preceding problem $=a$ shillings, the price of the poorest$=b$ shillings, the number of gallons in the mixture$=n$, and the price of the mixture$=c$. How many gallons of each kind must he use?

Ans. $\dfrac{(a-c)n}{a-b}$ gallons of the poorest, and $\dfrac{(c-b)n}{a-b}$ of the other.

59. A wine merchant has 40 bottles of wine, one of which he sells for 7 shillings; since he deems this price too high for his customers, he wishes to add as much water as will enable him to sell a bottle of the mixed wine for 6 shillings. How many bottles of water must he add to it? *Ans.* $6\frac{2}{3}$ bottles.

60. A father, who has three children, bequeaths his property by will in the following manner: To the eldest son he leaves a sum a, together with the nth part of what remains; to the second he leaves a sum $2a$, together with the nth part of what remains after the portion of the eldest and $2a$ have been subtracted from the estate; to the third he leaves a sum $3a$, together with the nth part of what remains after the portions of the two other sons and $3a$ have been subtracted. The property is found to be entirely disposed of by this arrangement. What was the amount of the property? *Ans.* $\dfrac{(6n^2-4n+1)a}{(n-1)^2}$.

61. A person purchases goods for \$4500, which he is to pay for at the expiration of a year. He afterwards agrees to pay the seller \$1500 cash, and the remaining \$3000 he is to pay in four equal payments of \$750, at equal times. What period must be fixed upon for these equal payments to be made, so that neither party may be a loser? *Ans.* $7\frac{1}{5}$ months.

62. During a panic there was a run on two bankers, A and B. B stopped payment at the end of three days, in consequence of which the alarm increased, and the daily demand for cash on A being trebled, A failed at the end of two more days. But if A and B had joined their capitals, they might both have stood the run, as it was at first, for 7 days, at the end of which time B would have been indebted to A \$4000. What was the daily demand for cash on A's bank at first? *Ans.* \$2000.

63. A waterman finds by experience that he can, with the advantage of a common tide, row down a river from A to B, which is 18 miles, in an hour and a half, and that to return from B to A, against an equal tide, though he rows back along the shore, where the stream is only three fifths as strong as in the middle, takes him just two hours and a quarter. From these data it is required to find at what rate per hour the tide runs in the middle, where it is strongest.

Ans. At the rate of $2\frac{1}{2}$ miles per hour.

64. As A and B were going to school, A first shot an arrow in the direction in which they were going, which B took up and shot forward; and so on alternately till the arrow had passed exactly from one milestone to another; when it appeared that A had shot the arrow 8 times and B 7 times. Some time afterwards, A and B were on the opposite banks of a river, the breadth of which they wished to ascertain; A first shot the arrow across the river, and it flew 13 yards beyond the bank on which B stood; B then took it up, and from the place where it had fallen, shot it back across the river; it now fell $9\frac{2}{7}$ yards beyond the bank on which A stood. Required the breadth of the river?

Ans. 100 yards.

EQUATIONS WHICH INVOLVE TWO OR MORE UNKNOWN QUANTITIES.

(75.) In the solution of many problems in simple equations, it is necessary to employ two or more unknown quantities. In such cases, the number of equations must always equal the number of unknown quantities employed. For, in the equation, $x=y+4$, any value may be assigned to y, which value augmented by 4 will furnish the corresponding value of x. Hence, in a simple equation, consisting of two unknown quantities, the unknown quantities may have an infinite number of values. Such an equation is called an indeterminate equation, and x, the value of which depends on that of y, is said to be a function of y. But if we have two independent equations involving two unknown quantities, each of the unknown quantities admits of only one value, and these values may be found.

The method of operating upon two or more simple equations, so as to obtain one equation, containing only one unknown quantity, is called *elimination.* There are three methods of elimination; namely, *by addition and subtraction*, *by substitution*, and *by comparison.*

FIRST METHOD.

By Addition and Subtraction.

(76.) Let it be required to find such values of x and y as will satisfy each of the following equations:

$$2x+3y=13 \qquad (1)$$
$$5x+4y=22 \qquad (2)$$

Multiply equation (1) by 4, and equation (2) by 3, and they become

$$8x+12y=52 \qquad (3)$$
$$\text{and } 15x+12y=66 \qquad (4)$$

Eq. (4)—Eq. (3)* gives $7x=14$ (5)

Whence $x=2$ (6)

$\therefore\ 8x=16$ (7)

* NOTE.—"Eq. (4) — eq. (3)" reads equation (4) minus equation (3). The abbreviation *eq.* stands for the word equation.

By substituting 16 for $8x$ in equation (3), it becomes

$$16+12y=52 \qquad (8)$$
$$\therefore \quad 12y=52-16 \qquad (9)$$
$$\text{Or} \quad 12y=36 \qquad (10)$$
$$\therefore \quad y=3 \qquad (11)$$

We might have first eliminated x; thus, multiply equation (1) by 5, and equation (2) by 2, and they become

$$10x+15y=65 \qquad (3)$$
$$10x+8y=44 \qquad (4)$$

Eq. (3)—eq. (4) gives $\quad 7y=21 \qquad (5)$

$$\therefore \quad y=3 \qquad (6)$$
$$\text{and} \quad 8y=24 \qquad (7)$$

Hence, eq. (4) may be written $\quad 10x+24=44 \qquad (8)$

$$\therefore \quad 10x=20, \text{ and } x=2 \qquad (9)$$

Before commencing the process of elimination, unite all the similar terms, and all the unknown quantities in each of the equations into the first member. If any of the equations have fractions, they may be cleared of fractions before trănsposing and uniting terms. It is, however, frequently advisable to transpose and unite some of the terms before clearing the equations of fractions. In reducing some equations it is better not to clear them of fractions. The student must rely on his own ingenuity. In the following equations transpose the 16 and the 18 before clearing the equation of fractions.

$$\frac{2x-y}{2}+14=18 \qquad (1)$$
$$\frac{2y+x}{3}+16=19 \qquad (2)$$

By transposing 14 and 16, these equations become,

$$\frac{2x-y}{2}=4 \qquad (3)$$
$$\frac{x+2y}{3}=3 \qquad (4)$$

Eq. (3) multiplied by 4 gives $\quad 4x-2y=16 \qquad (5)$

Eq. (4) " 3 gives $\quad x+2y=9 \qquad (6)$

Eq. (5)+eq. (6) gives $\quad 5x=25 \qquad (7)$

$$\therefore \quad x=5 \qquad (8)$$

In obtaining the values of x and y from the following equations, we do not clear them of fractions.

$$\frac{147}{x}-\frac{147}{y}=28 \qquad (1)$$

$$\frac{17}{x}+\frac{56}{y}=\frac{41}{3} \qquad (2)$$

Eq. (1) divided by 7 gives $$\frac{21}{x}-\frac{21}{y}=4 \qquad (3)$$

Eq. (2) multiplied by 21 gives $$\frac{17\times 21}{x}+\frac{56\times 21}{y}=287 \qquad (4)$$

Eq. (3) multiplied by 17 gives $$\frac{17\times 21}{x}-\frac{17\times 21}{y}=68 \qquad (5)$$

Eq. (4)—eq. (5) gives $$\frac{73\times 21}{y}=219 \qquad (6)$$

Eq. (6) divided by 219 gives $$\frac{7}{y}=1 \qquad (7)$$

$$\therefore \quad y=7 \qquad (8)$$

Substituting in (3) $$x=3 \qquad (9)$$

(**77.**) From what has been done, we see that an unknown quantity may be eliminated from two equations by applying the following

RULE.

Make the co-efficients of the unknown quantity to be eliminated, the same in each of the equations; then, if the signs of the unknown quantity are alike, subtract one equation from the other, but if the signs are unlike, add one equation to the other.

In order to obtain the values of three unknown quantities from three equations, we can eliminate one of the unknown quantities from the first and second equations, and then eliminate the same unknown quantity from the second and third equations, by the above rule. We shall thus have two equations, containing only two unknown quantities, from which we may obtain one equation containing only one unknown quantity, the value of which may be found. By retracing the several steps in the operation, it will be easy to find the values of the other two unknown quantities.

For the purpose of eliminating the unknown quantities, we may combine the equations in several ways, and the nature of the example should decide in what manner the unknown quantities should be eliminated. It is obvious that this method of eliminating may be applied, when we have m equations containing m unknown quantities. The student will become familiar with this method of elimination by studying the solutions of the following examples:

$$\text{Given} \left\{\begin{array}{ll} 3x+2y+\ z=16 & (1) \\ 2x+2y+2z=18 & (2) \\ 2x+2y+\ z=14 & (3) \end{array}\right\} \text{to find the values of } x, y, \text{ and } z.$$

$$\begin{array}{lrl} \text{Eq. (2)} - \text{eq. (3) gives } z= & 4 & (4) \\ \text{Eq. (1)} - \text{eq. (3) gives } x= & 2 & (5) \\ \text{Whence} \qquad 3x= & 6 & (6) \end{array}$$

Substituting these values of $3x$ and z, in eq. (1), and we have

$$\begin{array}{lrl} & 6+2y+4=16 & (7) \\ \text{By transposing} & 2y=\ 6 & (8) \\ \text{Whence} & y=\ 3 & (9) \end{array}$$

$$\text{Given} \left\{\begin{array}{ll} 5x-6y+4z=15 & (1) \\ 7x+4y-3z=19 & (2) \\ 2x+\ y+6z=46 & (3) \end{array}\right\} \text{to find } x, y, \text{ and } z.$$

$$\begin{array}{lrl} \text{Eq. (3) multiplied by 6 gives} & 12x+\ 6y+36z=\ 276 & (4) \\ \text{Eq. (1)+eq. (4) gives} & 17x+40z=\ 291 & (5) \\ \text{Eq. (3) multiplied by 4 gives} & 8x+\ 4y+24z=\ 184 & (6) \\ \text{Eq. (6)} - \text{eq. (2) gives} & x+27z=\ 165 & (7) \\ \text{Eq. (7) multiplied by 17 gives} & 17x+459z=2805 & (8) \\ \text{Eq. (8)} - \text{eq. (5) gives} & 419z=2514 & (9) \\ \text{Whence} & z=6 & (10) \end{array}$$

Substitute this value of z in eq. (7), and we have

$$\begin{array}{lrl} & x+162=165 & (11) \\ \text{By transposing} & x=3 & (12) \end{array}$$

By substituting the values of x and z in (3), we have

$$\begin{array}{lrl} & 6+y+36=46 & (13) \\ \text{By transposing} & y=\ 4 & (14) \end{array}$$

SECOND METHOD.

Elimination by Substitution.

(78.) Let it be required to find the values of x and y in the following equations:

$$2x+3y=13 \qquad (1)$$
$$5x+4y=22 \qquad (2)$$

By transposing $3y$ in eq. (1), and dividing by 2, we have

$$x=\frac{13-3y}{2} \qquad (3)$$

Whence $$5x=\frac{65-15y}{2} \qquad (4)$$

By substituting this value of $5x$ in eq. (2), we have

$$\frac{65-15y}{2}+4y=22 \qquad (6)$$

Or $$65-15y+8y=44 \qquad (7)$$

By transposing and uniting terms in (7), we have

$$-7y=-21 \qquad (8)$$

Whence $$y=3 \qquad (9)$$

Substituting this value in (3), $$x=\frac{13-3y}{2}=2 \qquad (10)$$

Hence, to eliminate by this method we have the following

RULE.

Find the value of one of the unknown quantities in one of the equations, in terms of the other unknown quantities, and substitute this value for the unknown quantity in each of the remaining equations, and thus make the number of equations and the number of unknown quantities one less. Proceed in this manner till a single equation is obtained containing but one unknown quantity, the value of which may then be found, and also the values of the other unknown quantities.

THIRD METHOD.

Elimination by Comparison.

(**79.**) Let it be required to find the values of x and y in the following equations:

$$2x+3y=13 \qquad (1)$$
$$5x+4y=22 \qquad (2)$$

By transposing $3y$ in (1) and dividing by 2, we have

$$x=\frac{13-3y}{2} \qquad (3)$$

By transposing $4y$ in (2), and dividing by 5, we have

$$x=\frac{22-4y}{5} \qquad (4)$$

Equating these two values of x, we have

$$\frac{13-3y}{2}=\frac{22-4y}{5} \qquad (5)$$

Eq. (5) multiplied by 10 gives $65-15y=44-8y$ (6)

By transposing $-7y=-21$ (7)

Whence $y=3$ (8)

Substituting this value of y in (3), we have

$$x=\frac{13-9}{2}=2 \qquad (9)$$

As another example, take the equations

$$2x+5y-3z=3 \qquad (1)$$
$$3x-4y+\ z=-2 \qquad (2)$$
$$5x-\ y+2z=9 \qquad (3)$$

By transposing $5y$ and $-3z$ in (1), and dividing by 2, we have

$$x=\frac{3-5y+3z}{2} \qquad (4)$$

By transposing $-4y$ and z in (2), and dividing by 3, we have

$$x=\frac{4y-z-2}{3} \qquad (5)$$

By transposing $-y$ and $2z$ in (3), and dividing by 5, we have

$$x=\frac{y-2z+9}{5} \qquad (6)$$

By equating these values of x, we have

$$\frac{4y-z-2}{3}=\frac{3-5y+3z}{2} \quad (7)$$

$$\text{And } \frac{4y-z-2}{3}=\frac{y-2z+9}{5} \quad (8)$$

Eq. (7) multiplied by 6 gives $8y-2z-4=9-15y+9z$ (9)

Eq. (8) " 15 " $20y-5z-10=3y-6z+27$ (10)

By transposing in (9) $23y-11z=13$ (11)

" " (10) $17y+z=37$ (12)

From equations (11) and (12) we readily obtain

$$z=\frac{23y-13}{11} \quad (13)$$

$$\text{And } z=37-17y \quad (14)$$

$$\text{Whence } \frac{23y-13}{11}=37-17y \quad (15)$$

$$\text{Or } 23y-13=407-187y \quad (16)$$

$$\text{By transposing } 210y=420 \quad (17)$$

$$\text{Hence } y=2 \quad (18)$$

By substituting this value of y in (14), we have

$$z=37-34=3 \quad (19)$$

By substituting the values of y and z in (5), we obtain

$$x=\frac{8-3-2}{3}=1 \quad (20)$$

From what has been done, we see that we may eliminate by this method by the following

RULE.

Find the values of one of the unknown quantities from each of the equations, as though the other unknown quantities were known; and then take one of these values and equate it with each of the other values. We shall thus form as many equations less one as there were at first, containing as many unknown quantities as there are new equations. Proceed in this manner till a single equation is obtained containing but one unknown quantity, from which its value may be found. By retracing the steps in the operation, the values of all the unknown quantities may be successively obtained.

EXAMPLES.

1. Given $\left\{\begin{array}{ll} 7x+5y=31 & (1) \\ 2x+4y=14 & (2) \end{array}\right\}$ to find the values of x and y.

$$\begin{array}{lrr}
\text{Eq. (1)+eq. (2) gives} & 9x+9y=45 & (3) \\
\text{Whence} & x+\ y=\ 5 & (4) \\
 & \therefore\ 2x+2y=10 & (5) \\
\text{Eq. (2)}-\text{eq. (5) gives} & 2y=\ 4\ \therefore y=2 & (6) \\
\text{Whence, by substituting,} & x=\ 3 & (7)
\end{array}$$

2. Given $\left\{\begin{array}{ll} \frac{1}{x}+\frac{1}{y}=\frac{1}{3} & (1) \\ \frac{1}{x}+\frac{1}{z}=\frac{1}{4} & (2) \\ \frac{1}{y}+\frac{1}{z}=\frac{1}{6} & (3) \end{array}\right\}$ to find the values of x, y, and z.

$$\begin{array}{lrr}
\text{Eq. (1)+eq. (2)+eq. (3) gives} & \frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{18}{24} & (4) \\
\text{Eq. (4) divided by 2 gives} & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{9}{24} & (5) \\
\text{Eq. (5)}-\text{eq. (1) gives} & \frac{1}{z}=\frac{1}{24} & (6) \\
\text{Eq. (5)}-\text{eq. (2) gives} & \frac{1}{y}=\frac{1}{8} & (7) \\
\text{Eq. (5)}-\text{eq. (3) gives} & \frac{1}{x}=\frac{5}{24} & (8)
\end{array}$$

From (6), (7), and (8) we readily find that

$$\left.\begin{array}{r} x=4\frac{4}{5} \\ y=8 \\ \text{and } z=24 \end{array}\right\} \text{Ans.}$$

3. Given $\left\{\begin{array}{ll} x(z+y)=14 & (1) \\ y(x+z)=18 & (2) \\ z(x+y)=20 & (3) \end{array}\right\}$ to find the value of x, y, and z.

Perform the multiplications indicated, and then add the three equations together, and divide their sum by 2.

	$xy+xz+yz=26$	(4)
Eq. (4)—eq. (1) gives	$yz=12$	(5)
Eq. (4)—eq. (2) gives	$xz=8$	(6)
Eq. (4)—eq. (3) gives	$xy=6$	(7)
Eq. (5)$\times$eq. (6)$\times$eq. (7) gives	$x^2y^2z^2=576$	(8)
Extract the square root of (8)	$xyz=24$	(9)
Eq. (9) divided by eq. (5) gives	$x=2$	(10)
Eq. (9) divided by eq. (6) gives	$y=3$	(11)
Eq. (9) divided by eq. (7) gives	$z=4$	(12)

4. Given $\left\{\begin{array}{lr} x+ly+lz=p & (1) \\ mx+y+mz=q & (2) \\ nx+ny+z=r & (3) \end{array}\right\}$ to find the values of x, y, and z,—l, m, n, p, q, and r, are known quantities.

Let $x+y+z=s$ (A); then the equations may be made to assume this form, by substituting s, adding and subtracting lx to the first member of (1), my to the first member of (2), and nz to the first member of (3).

$$(1-l)x+ls=p \qquad (4)$$

$$(1-m)y+ms=q \qquad (5)$$

$$(1-n)z+ns=r \qquad (6)$$

From (4), (5), and (6), we readily obtain (7), (8), and (9),

$$x+\frac{l}{1-l}s=\frac{p}{1-l} \qquad (7)$$

$$y+\frac{m}{1-m}s=\frac{q}{1-m} \qquad (8)$$

$$z+\frac{n}{1-n}s=\frac{r}{1-n} \qquad (9)$$

Eq. (7)+eq. (8)+eq. (9) gives $s+\left(\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}\right)s=$
$\frac{p}{1-l}+\frac{q}{m-1}+\frac{r}{1-n}$ (10)

Whence
$$s=\frac{\frac{p}{1-l}+\frac{q}{1-m}+\frac{r}{1-n}}{1+\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}} \qquad (11)$$

Substituting this value of s in (7), (8), and (9), and reducing, we obtain

$$x=\frac{p}{1-l}-\frac{l}{1-l}\left\{\frac{\frac{p}{1-l}+\frac{q}{1-m}+\frac{r}{1-n}}{1+\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}}\right\} \quad (12)$$

$$y=\frac{q}{1-m}-\frac{m}{1-m}\left\{\frac{\frac{p}{1-l}+\frac{q}{1-m}+\frac{r}{1-n}}{1+\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}}\right\} \quad (13)$$

$$z=\frac{r}{1-n}-\frac{n}{1-n}\left\{\frac{\frac{p}{1-l}+\frac{q}{1-m}+\frac{r}{1-n}}{1+\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}}\right\} \quad (14)$$

If the student will carefully examine these equations, commencing with (1), he will see how they can be successively deduced from each other. Such equations are called symmetrical equations. The equations in example (3) are symmetrical equations.

5. Given $\left\{\begin{array}{l} 5x+7y=43 \quad (1) \\ 11x+9y=69 \quad (2) \end{array}\right\}$ to find x and y.

Ans. $x=3$, $y=4$.

6. Given $\left\{\begin{array}{l} 8x-21y=33 \quad (1) \\ 6x+35y=177 \quad (2) \end{array}\right\}$ to find x and y.

Ans. $x=12$, $y=3$.

7. Given $\left\{\begin{array}{l} 7y-2x=1 \quad (1) \\ 20y+6x=38 \quad (2) \end{array}\right\}$ to find x and y.

Ans. $y=1$, $x=3$.

8. Given $\left\{\begin{array}{l} 12x+32y=340 \quad (1) \\ 8x+24y=254 \quad (2) \end{array}\right\}$ to find x and y.

Ans. $x=1$, $y=10\frac{1}{4}$.

9. Given $\left\{\begin{array}{ll} x(z+y)=5a & (1) \\ y(x+z)=7a & (2) \\ z(x+y)=4a & (3) \end{array}\right\}$ to find x, y, and z.

Ans. $x^*=\frac{2}{3}\sqrt{3a}$, $y=2\sqrt{3a}$, and $z=\frac{1}{2}\sqrt{3a}$.

10. Given $\left\{\begin{array}{ll} \frac{4x}{x^2}+\frac{5y}{y^2}=\frac{9}{y}-1 & (1) \\ \frac{5}{x}+\frac{4}{y}=\frac{7}{x}+\frac{3}{2} & (2) \end{array}\right\}$ to find x and y.

Ans. $x=4$, $y=2$.

11. Given $\left\{\begin{array}{ll} u+x+y=13 & (1) \\ u+x+z=17 & (2) \\ u+y+z=18 & (3) \\ x+y+z=21 & (4) \end{array}\right\}$ to find x, y, z, and u.

Ans. $x=5$, $y=6$, $z=10$, $u=2$.

12. Given $\left\{\begin{array}{ll} 2x=y+z+u & (1) \\ 3y=x+z+u & (2) \\ 4z=x+y+u & (3) \\ u=x-14 & (4) \end{array}\right\}$ to find x, y, z, and u.

Ans. $x=40$, $u=26$, $y=30$, $z=24$.

13. Given $\left\{\begin{array}{ll} x+22=y+z & (1) \\ y+22=2x+2z & (2) \\ z+22=3x+3y & (3) \end{array}\right\}$ to find x, y, and z.

Ans. $x=2$, $y=10$, $z=14$.

14. Given $\left\{\begin{array}{ll} 4x-4y-4z=a & (1) \\ 6y-2x-2z=a & (2) \\ 7z-y-x\ \ =a & (3) \end{array}\right\}$ to find x, y, and z.

Ans. $x=\frac{13}{8}a$, $y=\frac{7}{8}a$, $z=\frac{1}{2}a$.

15. Given $\left\{\begin{array}{ll} \frac{5}{3}x+\frac{y}{2}=12-\frac{3y}{4}+\frac{1}{12} & (1) \\ \frac{y}{6}-\frac{x}{2}=4\frac{1}{6}-2x & (2) \end{array}\right\}$ to find x and y.

Ans. $x=2$, and $y=7$.

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* NOTE.—See Art. , Ex.

16. Given $\left.\begin{cases} 3xy=10x+8y & (1) \\ 2xy=26y-45x & (2) \end{cases}\right\}$ to find x and y.

Ans. $x=4$, $y=10$.

17. Given $\left.\begin{cases} xy^2+x\ -2xy\ =16 & (1) \\ xy^3+xy-2xy^2=48 & (2) \end{cases}\right\}$ to find x and y.

Ans. $x=4$, $y=3$.

18. Given $\left.\begin{cases} 2x+2y+\ 4z=\ 14 & (1) \\ 3x+5y+42z=\ 63 & (2) \\ 5x+3y-36z=-17 & (3) \end{cases}\right\}$ to find x, y and z.

Ans. $x=2$, $y=3$, $z=1$.

19. Given $\left.\begin{cases} 3x+6y+1=\dfrac{6x^2-24y^2+130}{2x-4y+3} \\ 3x=\dfrac{9xy-110}{3y-4}+\dfrac{151-16x}{4y-1} \end{cases}\right\}$ to find x and y.

Ans. $x=9$, $y=2$.

20. Given $\left.\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a} & (1) \\ \dfrac{1}{x}+\dfrac{1}{z}=\dfrac{1}{b} & (2) \\ \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{c} & (3) \end{cases}\right\}$ to find x, y, and z.

Ans. $x=\dfrac{2abc}{ac-ab+bc}$, $y=\dfrac{2abc}{ab-ac+bc}$, $z=\dfrac{2abc}{ac+ab-bc}$.

21. Given $\left.\begin{cases} \dfrac{a}{b+y}=\dfrac{b}{3a+x} & (1) \\ ax+2by=c & (2) \end{cases}\right\}$ to find x and y.

Ans. $x=\dfrac{2b^2-6a^2+c}{3a}$, $y=\dfrac{3a^2-b^2+c}{3b}$.

22. Given $\left.\begin{cases} x^2+y^2=2a & (1) \\ \dfrac{x^4-y^4}{x^2+y^2}=2b\ * & (2) \end{cases}\right\}$ to find x and y.

Ans. $x=\sqrt{a+b}$, $y=\sqrt{a-b}$.

$* \quad \dfrac{(x^2+y^2)\times(x^2-y^2)}{x^2+y^2}=2b$

$\therefore \quad 2x^2=2a+2b$

$x=\sqrt{a+b}$

23. Given $\left\{\begin{array}{ll} 17x+23y=103 & (1) \\ 13x+\ \ 7y=\ \ 47 & (2) \end{array}\right\}$ to find x and y.

Ans. $x=2$, $y=3$.

24. Given $\left\{\begin{array}{ll} 2x+300=3y-150 & (1) \\ 9x-450=5y+500 & (2) \end{array}\right\}$ to find x and y.

Ans. $x=300$, $y=350$.

25. Given $\left\{\begin{array}{ll} bcx=cy-2b & (1) \\ b^2y+\dfrac{a(c^3-b^3)}{bc}=\dfrac{2b^3}{c}+c^3x & (2) \end{array}\right\}$ to find x and y.

Ans. $x=\dfrac{a}{bc}$, $y=\dfrac{a+2b}{c}$.

26. Given $\left\{\begin{array}{ll} 3xy=28-\ y^2 & (1) \\ 9x\ =21-3y & (2) \end{array}\right\}$ to find x and y.

Ans. $x=1$, $y=4$.

27. Given $\left\{\begin{array}{ll} 7x-2z+3u=17 & (1) \\ 4y-2z+\ \ t=11 & (2) \\ 5y-3x-2u=\ \ 8 & (3) \\ 4y-3u+2t=\ \ 9 & (4) \\ 3z+8u=33 & (5) \end{array}\right\}$ to find x, y, z, u, and t.

Ans. $x=2$, $y=4$, $z=3$, $u=3$, $t=1$.

28. Given $\left\{\begin{array}{ll} x+y=10 & (1) \\ x^2-y^2=40 & (2) \end{array}\right\}$ to find x and y.

Ans. $x=7$, $y=3$.

29. Given $\left\{\begin{array}{ll} 5x+2y+2z=24 & (1) \\ 4x+7y+5z=49 & (2) \\ x+\ \ y+3z=17 & (3) \end{array}\right\}$ to find x, y, and z.

Ans. $x=2$, $y=3$, $z=4$.

ELIMINATION BY INDETERMINATE MULTIPLIERS.

(80.) Let it be required to find the values of x and y from he following equations:

$$3x-2y=10 \qquad (1)$$
$$4x+3y=36 \qquad (2)$$

Multiply the second equation by the indeterminate quantity, m, and we have

$$4mx+3my=36m \quad (3)$$

Add equation (3) to equation (1), and we have

$$4mx+3x+3my-2y=36m+10 \quad (4)$$

$$\text{Or, } (4m+3)x+(3m-2)y=36m+10 \quad (5)$$

Since m may have any value, we may give to it such a value as will cause the co-efficient of y in equation (5) to become nothing. Hence, in order to eliminate y, we may assume

$$3m-2=0 \quad (6)$$

$$\text{Whence, } \quad m=\tfrac{2}{3} \quad (7)$$

Since $3m-2=0$, equation (5) becomes

$$(4m+3)x=36m+10 \quad (8)$$

$$\text{Or, } \quad x=\frac{36m+10}{4m+3} \quad (9)$$

Substitute the value of m in equation (7) in the place of m in equation (9), and we readily obtain

$$x=6 \quad (10)$$

In a similar way we may find that $\quad y=4 \quad (11)$

(**81.**) We will now, by the aid of this principle, determine the values of x, y, and z, from the following equations, in which a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, are the co-efficients of the unknown quantities, and m_1, m_2, m_3, are the absolute terms.

$$\left.\begin{array}{ll} a_1x+b_1y+c_1z=m_1 & (1) \\ a_2x+b_2y+c_2z=m_2 & (2) \\ a_3x+b_3y+c_3z=m_3 & (3) \end{array}\right\} (A)$$

Multiply equation (2) by r, and equation (3) by s, and equations (A) become

$$\left.\begin{array}{ll} a_1\,x+b_1\,y+c_1\,z=m_1 & (4) \\ a_2rx+b_2ry+c_2rz=m_2r & (5) \\ a_3sx+b_3sy+c_3sz=m_3s & (6) \end{array}\right\} (B)$$

Assume $\left.\begin{array}{l} b_2r+b_3s=-b_1 \\ \text{and } c_2r+c_3s=-c_1 \end{array}\right\}$ Whence $\left\{\begin{array}{l} r=\dfrac{-c_3b_1+b_3c_1}{b_2c_3-b_3c_2} \\ s=\dfrac{-b_2c_1+b_1c_2}{b_2c_3-b_3c_2} \end{array}\right\}$ (D)

The values of r and s may be found from the equations, $b_2r+b_3s=-b_1$, and $c_2r+c_3s=-c_1$, by applying the rules of either of the three common methods of elimination, or by the method of indeterminate multipliers.

By taking the sum of equations (B), and recollecting that we have assumed that the sum of the co-efficient of y in equations (5) and (6), is equal to the co-efficients of y in equation (1), taken with a contrary sign, and that we have made a similar assumption in respect to the co-efficients of z in those equations, we have

$$a_1x+a_2rx+a_3sx=m_1+m_2r+m_3s \qquad (7)$$

Whence,
$$x=\frac{m_1+m_2r+m_3s}{a_1+a_2r+a_3s} \qquad (8)$$

In the place of r and s in this last equation, put their values, as found in equations (D), and we have

$$x=\frac{m_1+m_2\dfrac{-b_1c_3+b_3c_1}{b_2c_3-b_3c_2}+m_3\dfrac{-b_2c_1+b_1c_2}{b_2c_3-b_3c_2}}{a_1+a_2\dfrac{-b_1c_3+b_3c_1}{b_2c_3-b_3c_2}+a_3\dfrac{-b_2c_1+b_1c_2}{b_2c_3-b_3c_2}} \qquad (9)$$

Reducing this value of x to a more simple form by multiplying both numerator and denominator by $b_2c_3-b_3c_2$, and we have

$$x=\frac{m_1b_2c_3-m_1b_3c_2-m_2b_1c_3+m_2b_3c_1-m_3b_2c_1+m_3b_1c_2}{a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2b_3c_1-a_3b_2c_1+a_3b_1c_2} \qquad (10)$$

By arranging the terms in the numerator and denominator in a different order, we have

$$\left.\begin{array}{ll} x=\dfrac{m_1b_2c_3+m_2b_3c_1+m_3b_1c_2-m_1b_3c_2-m_2b_1c_3-m_3b_2c_1}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1} & (11) \\ y=\dfrac{m_2a_1c_3+m_3a_2c_1+m_1a_3c_2-m_3a_1c_2-m_1a_2c_3-m_2a_3c_1}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1} & (12) \\ z=\dfrac{m_3a_1b_2+m_1a_2b_3+m_2a_3b_1-m_2a_1b_3-m_3a_2b_1-m_1a_3b_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1} & (13) \end{array}\right\} \text{(E)}$$

The values of y and z were obtained in the same manner. By carefully examining equations (E), and also equations (1), (2), (3), the intelligent student will find it easy to point out in what manner the values of x, y, and z may be obtained from the co-efficients and the absolute terms. It is obvious, that by giving particular values to the co-efficients and the absolute terms, we may, by the aid of equations (E), obtain the values of x, y, and z from any three simple equations involving these letters.

ON THE SOLUTION OF PROBLEMS WHICH REQUIRE TWO OR MORE UNKNOWN QUANTITIES.

1. A banker has two kinds of money; it takes a pieces of the first kind to make a crown, and b pieces of the second kind to make the same. Some one wishes to have c pieces for a crown. How many pieces of each must the banker give him?

Let x=the number of pieces that he uses of the 1st kind,
And y= " " " " 2d " .

$\frac{1}{a}$=the part of a crown which 1 piece of the 1st kind makes,*
$\frac{1}{b}$= " " " " 2d "
$\frac{x}{a}$= " " " x pieces 1st kind make,
$\frac{y}{b}$= " " " y " 2d "

Whence $$x+y=c \qquad (1)$$

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (2)$$

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* NOTE.—Since a pieces of the first kind make a crown, one piece of the first kind will make one ath part of a crown, or a part denoted by $\frac{1}{a}$, and x pieces will make x times $\frac{1}{a}$ part of a crown, or a part denoted by $\frac{x}{a}$. In the same way we find that y pieces of 2d kind will make a part of the crown denoted by $\frac{y}{b}$. These two parts added together make one crown.

Eq. (2)$\times ab$ gives $\quad bx+ay=ab \qquad (3)$

Eq. (1)$\times a$ gives $\quad ax+ay=ac \qquad (4)$

Eq. (1)$\times b$ gives $\quad bx+by=bc \qquad (5)$

Eq. (4)$-$eq. (3) gives $\quad ax-bx=ac-ab \qquad (6)$

Eq. (3)$-$eq. (5) gives $\quad ay-by=ab-bc \qquad (7)$

$$\left.\begin{array}{ll} \text{From (6)} \quad x=\dfrac{ac-ab}{a-b}=\dfrac{a(c-b)}{a-b} & (8) \\ \text{From (7)} \quad y=\dfrac{ab-bc}{a-b}=\dfrac{b(a-c)}{a-b} & (9) \end{array}\right\} \text{Ans.}$$

2. According to Vitruvius, the crown of Hiero, king of Syracuse, weighed 20 lbs., and lost $1\frac{1}{4}$ lbs., nearly, when weighed in water. Let it be assumed that it consisted of gold and silver only, and that 19.64 lbs. of gold lose 1 lb. in water, and 10.5 lbs. of silver, in like manner, lose 1 lb. How much gold, and how much silver, did this crown contain?

If we let x=the number of pounds of gold, and y=the number of pounds of silver, and make $20=c$, $19.64=a$, $10.5=b$, we shall find that the solution of this problem is similar to that of the preceding one. By reducing the equations which the problem readily furnishes, we shall find that $x=14.77$, $y=5.22$.

3. Some hours after a courier had been sent from A to B, which are 147 miles distant, a second was sent, who wished to overtake the first just as he entered B; in order to do this, he found that he must perform the journey in 28 hours less time than the first did. Now, the time in which the first travels 17 miles added to the time in which the second travels 56 miles is $13\frac{2}{3}$ hours. How many miles does each go per hour?

Let x=number of miles that the 1st travels per hour.
And y= " " 2d " "

$\dfrac{147}{x}$=number of hours which the 1st requires to travel 147 miles.

$\dfrac{147}{y}$= " " 2d " " "

$\dfrac{17}{x}$= " " 1st " " 17 miles.

$\dfrac{56}{y}$= " " 2d " " 56 "

By the conditions of the problem we have,

$$\frac{147}{x}-\frac{147}{y}=28 \qquad (1)$$

$$\frac{17}{x}+\frac{56}{y}=13\tfrac{2}{3}=\tfrac{41}{3} \qquad (2)$$

These equations are the same as those found on page 83. Hence, the first travels 3 miles per hour, and the second travels 7 miles per hour.

4. A certain number consists of three digits, the sum of which is 9. If 198 be subtracted from the number, the remainder is a number consisting of the same digits as the one sought, but in an inverted order; and if the number be divided by the left digit, the quotient is 108. What is the number?

Let x=the number of hundreds,
y= " " tens,
And z= " " units,
Then, $100x+10y+z$=the number.

By the conditions of the problem, we have the following equations:

$$x+y+z=9, \qquad (1)$$

$$100x+10y+z-198=100z+10y+x, \qquad (2)$$

$$\text{And } \frac{100x+10y+z}{x}=108 \qquad (3)$$

By transposing, &c. in (2)	$99x-99z=198$	(4)
From (3)	$-8x+10y+z=\ 0$	(5)
Eq. (1)−eq. (5) gives	$9x-9y=\ 9$	(6)
Eq. (6)÷9 gives	$x-y=\ 1$	(7)
Eq. (4)÷99 gives	$x-z=\ 2$	(8)
Eq. (1)+eq. (7) gives	$2x+z=\ 10$	(9)
Eq. (8)+eq. (9) gives	$3x=\ 12$	(10)
	$\therefore\ x=\ 4$	(11)
From (7),	$y=x-1=3$	(12)
From (8),	$z=x-2=2$	(13)

Hence, the number is 432.

5. Required two numbers whose sum is 70, and whose difference is 16. *Ans.* 43 and 27.

6. Required two numbers whose sum is a, and whose difference is b. *Ans.* $\frac{a+b}{2}$, and $\frac{a-b}{2}$.

7. Two purses, together, contain \$300. If you take \$30 from the first, and put them into the second, then there is the same number of dollars in each. How many dollars does each contain?
Ans. The first \$180, the second \$120.

8. A owes \$1200, B \$2500; but neither has enough to pay his debts. Lend me, said A to B, the 8th part of your fortune, and I shall be enabled to pay my debts. B answered, I can discharge my debts, if you will lend me the 9th part of your fortune. What was the fortune of each?
Ans. A's 900, B's 2400.

9. Five gamblers, A, B, C, D, E, throw dice upon the condition that he who has the lowest throw shall give all the rest the sum which they already have. Each gamester loses in turn, commencing with A, and at the end of the fifth game, all have the same sum, viz., \$32. How much had each at first?

10. A and B, jointly, have a fortune of \$9800. A invests the 6th part of his property in business, and B the 5th part of his, and each has then the same sum remaining. How much has each? *Ans.* A has \$4800, B \$5000.

11. A capitalist borrows \$8000 on favorable conditions, because he has an opportunity of lending \$23000 at such a rate per cent. that the interest which he receives exceeds that which he pays by \$905. On the same conditions he borrows \$9400, and then lends \$17500, and the interest which he receives exceeds that which he pays by \$539.50 yearly. At what rate per cent. did he borrow and lend money?
Ans. At $4\frac{1}{2}$ and $5\frac{1}{2}$ per cent.

12. Find two numbers of the following properties: When the one is multiplied by 2, the other by 5, and both products added together, the sum is 31; and if the first be multiplied by 7, and the second by 4, and the products added together, the sum is 68.

Ans. The first is 8, and the second is 3.

13. A person has two kinds of goods. Now, 8 lbs. of the first and 9 lbs. of the second cost \$18.46; further, 20 lbs. of the first and 16 of the second cost \$36.40. How much does the pound of each article cost? *Ans.* 62 cents, and \$1.50.

14. A person has two large pieces of iron, whose weight is required. It is known that $\frac{2}{5}$ of the first piece weighs 96 lbs. less than $\frac{3}{4}$ of the other piece; and that $\frac{5}{8}$ of the other piece weighs exactly as much as $\frac{4}{9}$ of the first. How much did each of these pieces weigh?

Ans. The first weighs 720, the second 512 lbs.

15. A cistern containing 210 buckets, may be filled by 2 pipes. By an experiment, in which the first pipe was open 4, and the second 5 hours, 90 buckets of water were obtained. By another experiment, when the first was open 7, and the other $3\frac{1}{2}$ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour? And in what time will the cistern be full, when the water flows from both pipes at once?

Ans. The first pipe discharges 15, and the second 6 buckets; it will require 10 hours for them to fill the cistern.

16. It is required to find a fraction such, that if 3 be subtracted from the numerator and denominator, it is changed into $\frac{1}{4}$, and if 5 be added to the numerator and denominator, it becomes $\frac{1}{2}$. What is the fraction? *Ans.* $\frac{7}{19}$.

17. A wine merchant has two kinds of wine. If he mix 3 gallons of the poorer wine with 5 of the better, the mixture is worth \$1 per gallon; but if he mixes $3\frac{1}{4}$ gallons of the poorer with $8\frac{3}{4}$ gallons of the better, the mixture is worth $\$1.03\frac{1}{3}$ per gallon. What does each wine cost per gallon?

Ans. The better \$1.12, the poorer \$0.80.

18. There is a fraction such that if 1 be added to the numerator, its value $=\frac{1}{3}$, and if 1 be added to the denominator its value $=\frac{1}{4}$. What is the fraction? *Ans.* $\frac{4}{15}$.

19. A person has a sum of money which he places out at interest. Another person possesses \$10000 more than the first, and placing his capital out at interest at a rate one per cent. higher than that of the first, he had an income greater by \$800. A third person possesses \$15000 more than the first, and placing his capital out at interest at a rate 2 per cent. higher than that of the first, he has an income greater by \$1500. Required the capital that each had, and the rates per cent.

Ans. $\begin{cases} \text{Capitals,} & \$30000, & \$40000, & \$45000. \\ \text{Rates of interest,} & 4, & 5, & 6, \end{cases}$

20. It is found by experiment that 37 lbs. of tin lose 5 lbs. when weighed in water, and that 23 lbs. of lead lose 2 lbs. in water. A composition of tin and lead, weighing 120 lbs., loses 14 lbs. in water. How much does this composition contain of each metal? *Ans.* 74 lbs. tin, and 46 lbs. lead.

21. A given piece of metal, which weighs p pounds, loses a pounds in water. This piece, however, is composed of two other metals, A and B; of these we know that p pounds of A lose b pounds in water, and p pounds of B lose c pounds. How many pounds of each metal does this piece contain?

Ans. $\frac{(c-a)p}{c-b}$ lbs. of A, and $\frac{(a-b)p}{c-b}$ lbs. of B.

22. Three masons, A, B, C, are to build a wall. A and B, jointly, can build the wall in 12 days; B and C can accomplish it in 20 days, and A and C in 15 days. How many days would each require to build the wall, and in what time will they finish it, if all three work together?

Ans. A requires 20 days, B 30, and C 60;
and all three require 10 days.

23. Three laborers are employed on a certain work. A and B, jointly, can complete the work in a days; A and C require b days, B and C require c days. What time does each one, working alone, require to accomplish the work, on the condition that each one, under all circumstances, does the same quantity of work? And in what time would they finish it, if they all three worked together?

Ans. A requires $\frac{2abc}{bc+ac-ab}$ days, B $\frac{2abc}{bc+ab-ac}$ days, and C $\frac{2abc}{ab+ac-bc}$ days.

Jointly, they require $\frac{2abc}{ab+ac+bc}$ days.

24. A, B, C, together, possess \$1820. If B gives A \$200 of his money, then A will have \$160 more than B; but if B receives \$70 from C, then both will have the same sum. How much has each? *Ans.* A \$400, B \$640, C \$780.

25. A and B possess, together, a fortune of \$570. If A's for tune were 3 times, and B's 5 times as great as each really is, they would have together \$2350. What was each one's fortune?

Ans. A's \$250, B's \$320.

26. B has lent out at interest \$12600 more than A, and obtains 1 per cent. more for his money, and his yearly interest is \$730 more than A's. C has lent \$3000 more than A, at 2 per cent. higher interest, and his yearly interest is \$380 more than A's. How much has each lent? And at what interest?

Ans. A \$10000, B \$22600, C \$13000.
A at 4 per cent., B at 5, and C at 6.

27. When the first of two numbers is increased by a, it becomes m times as great as the second; but when the second is increased by b it becomes n times as great as the first. How are these numbers expressed?

Ans. The first $=\frac{a+mb}{mn-1}$, the second $=\frac{b+na}{mn-1}$.

28. A work is to be printed, so that each page may contain a certain number of lines, and each line a certain number of letters. If we wished to have each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more on each page; but if we wished to have 2 lines less in a page, and 3 letters less in each line, then each page would contain 145 letters less. How many lines are there in each page? And how many letters in each line?

Ans. 29 lines in each page, and 32 letters in a line.

29. A father said to his son, "six years ago I was $3\frac{1}{3}$ times as old as you; but three years *hence* I shall be $2\frac{1}{6}$ times as old as you." What is the age of each?

Ans. The father 36, the son 15 years.

30. After A had won 4 shillings of B, he had only half as many shillings as B had left. But had B won 6 shillings of A, then he would have had three times as many as A would have had left. How many had each? *Ans.* A 36, B 84.

31. A person has two snuff boxes. If he puts \$8 into the first, then it is half as valuable as the second; but if he takes out \$8 from the first, and puts them into the second, then the latter is three times as valuable as the former. What is the value of each?

Ans. The first \$48, second \$112.

32. A, B, C compare their fortunes. A says to B, give me \$700 of your money, and I shall have twice as much as you retain; B says to C, give me \$1400, and I shall have three times as much as you have remaining; C says to A, give me \$420, and I shall then have five times as much as you retain. How much has each? *Ans.* A \$980, B \$1540, C \$2380.

33. The sum of two numbers is 13, and the difference of their squares is 39. What are the numbers?

Ans. 5 and 8.

34. The sum of two numbers is $=a$, and the greater number is n times the smaller. What are the numbers?

Ans. $\dfrac{a}{n+1}$, and $\dfrac{na}{n+1}$.

35. A merchant mixed tea which cost him 45 cents per pound, with some that was worth 75 cents per pound, and sold the mixture for \$24.30, and by this means gained 20 per cent. How many pounds of each kind were there, the whole number of pounds being 35?

Ans. 20 lbs. at 45 cents, and 15 lbs. at 75 cents.

36. A certain number, which has two digits, is equal to 8 times the sum of its digits, and if 45 be subtracted from the number, its digits will be inverted. What is the number?

Ans. 72.

37. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time would each do it separately?

Ans. A in 24 days, and B in 48 days.

38. There is a cistern, into which water is admitted by three pipes, two of which are exactly of the same dimensions. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal pipes be stopped, seven-ninths of the cistern is filled in 10 hours and 40 minutes. In how many hours would each pipe fill the cistern?

Ans. Each of the equal ones in 32 hours, and the other in 24 hours.

39. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each?

Ans. 21 crowns, and 63 guineas.

40. A rectangular bowling-green having been measured, it was observed, that if it were 5 yards broader, and 4 yards longer, it would contain 116 yards more; but if it were 4 yards broader, and 5 yards longer, it would contain 113 yards more. Required the length and breadth?

Ans. 12 yards long, and 9 yards wide.

41. A Corn-factor mixes wheat flour, which costs him 10 shillings a bushel, with barley flour, which costs him 4 shillings a bushel, in such proportion as to gain $43\frac{3}{4}$ per cent., by selling the mixture at 11 shillings per bushel. Required the proportion.

Ans. 14 bushels of wheat flour to 9 of barley.

42. Some smugglers discovered a cave which would exactly hold the cargo of their boat, viz., 13 bales of cotton, and 33 casks of rum. Whilst they were unloading, a custom-house cutter coming in sight, they sailed away with 9 casks and 5 bales, leaving the cave two-thirds full. How many bales or casks would it hold?

Ans. 24 bales, or 72 casks.

43. A countryman, being employed to drive a flock of geese and turkeys to New York, in order to distinguish his own from any he might meet on the way, pulled three feathers out of the tail of each turkey, and one out of the tail of each goose, and upon counting them, found that the number of turkey's feathers exceeded twice those of the geese by 15. Having bought 10 geese, and sold 15 turkeys by the way, he was surprised to find, as he drove them into his employer's yard, that the number of geese was equal to seven-thirds the number of turkeys. Required the number of each at first.

Ans. 45 turkeys, and 60 geese.

44. Round two wheels, whose circumferences are as 5 to 3, two ropes are wrapped, whose difference exceeds the difference of the circumferences by 280 yards. Now the longer rope applied to the larger wheel, wraps around it a certain number of times, greater by 12 than the shorter round the smaller wheel; and if the larger wheel turns round 3 times as quick as the other, the ropes will be discharged at the same time. Required the lengths of the ropes, and the circumferences of the wheels.

Ans. The lengths of the ropes are 360 and 72 yards, and the circumferences of the wheels are 20 and 12 yards.

45. A and B agree to reap a field of wheat in 12 days. The times in which they could severally reap an acre are as 2 to 3. After some time, finding themselves unable to finish it in the stipulated time, they called in C to help them, whose rate of working was such, that if he had wrought with them from the beginning, it would have been finished in 9 days. Also the times in which he could severally have reaped the field with A alone, and B alone, are in the proportion of 7 to 8. When was C called in?

Ans. After 6 days.

INTERPRETATION OF NEGATIVE RESULTS.

(82.) Algebraical formulas can convey no distinct idea to the mind, unless they represent numerical calculations, which can be actually performed. Thus, the expression $a-b$ indicates an absurdity, when b is greater than a, since it is impossible to subtract a greater quantity from a smaller one. As the solution of problems sometimes presents this difficulty, it will be necessary to show what signification ought to be attached to such a value of the unknown quantity.

Every Equation of the first degree may be reduced to the following form, in which all the signs are positive,

$$ax+n=cx+m \qquad (1)^*$$

Subtracting $cx+n$ from each member, we have,

$$ax-cx=m-n \qquad (2)$$

$$\text{Whence,} \quad x=\frac{m-n}{a-c} \qquad (3)$$

This being premised, three cases present themselves:

CASE I.

When $m>n$, and $a>c$.

CASE II.

When $m>n$, and $a<c$, or $m<n$, and $a>c$; that is, when only one of the conditions in Case I. *holds good.*

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* Note.—This may be done by transposing the negative terms to the other members.

CASE III.

When $m<n$, *and* $a<c$.

In the first case, it is plain that the value of x is positive, and that the solution of the problem gives rise to no absurdity.

In the second case, one of the subtractions, $m-n$, $a-c$ is impossible. If $m<n$ and $a>c$, for example, equation (1) must involve an absurdity, since, in this case, the two terms, ax and n, in the first member, are respectively greater than the two terms cx and m, in the second member. Now, as equation (1) may be regarded as truly representing the conditions of a problem, the conditions themselves must be absurd, whenever they give an equation, the solution of which presents this case.

In the third case, both of the subtractions, $m-n$ and $a-c$, are impossible, but we must not, for this reason, conclude that equation (1) presents an absurdity. For, let it be observed, that in obtaining the value of x, we subtracted $cx+n$ from each member, an operation which is impossible, since this quantity is greater than either the first or second member. To find the value of x, we will subtract $ax+m$ from each member of equation (1), and we then have,

$$n-m=cx-ax, \qquad (4)$$

$$\text{Whence,} \quad x=\frac{n-m}{c-a} \qquad (5)$$

The value of x in equation (5) may be derived from the value of x in equation (3), by changing the signs of both numerator and denominator of the fraction, which expresses the value of x in this equation. Now as the quotient of a negative quantity divided by a negative quantity is positive, we may change the signs of both numerator and denominator of the fraction, which expresses the value of x in equation (5), without changing its value from positive to negative. Hence equation (5) may be written,

$$x=\frac{-(n-m)}{-(c-a)}=\frac{m-n}{a-c} \qquad (6)$$

Hence we see that the value of x in equation (6), which is the true value of x in Case III., is the same as the value of x in equation (3).

One of the chief advantages which algebra presents, is to obtain formulas which shall include all the conditions of a problem. This end may be accomplished by establishing the convention, *that the same operations shall be performed on isolated negative quantities, as if they were connected with other magnitudes.* For example, in the expression $a+b-d$, we could not subtract d from b if d were greater than b, and we should write the expression in this form, $a-(d-b)$; b may now be subtracted from d, and the difference taken from a. If a does not exist, we may still write $b-d$ in the form, $-(d-b)$ when $b<d$.

We may, in the second case, write the value of x in this form, $-\frac{m-n}{c-a}$ when $m>n$, and $a<c$, and in this form, $-\frac{n-m}{a-c}$, when $m<n$, and $a>c$. Each of these values of x is negative. We conclude then, that

Every negative solution denotes some absurdity in the enunciation of the problem proposed.

For the purpose of showing the use of a negative solution in pointing out the error in the enunciation of a problem, we will propose the following problem:

A father is now 48 years old, and his son is 15. How many years must elapse, in order that the age of the father may be 4 times the age of the son?

Let x = the number of years that must elapse.

By the conditions of the problem, we have

$$48+x=4\times(15+x) \qquad (1)$$

Or, $$48+x=60+4x \qquad (2)$$

Whence, $$x=-4 \qquad (3)$$

This negative result indicates that an error was made in the enunciation of the problem. We can correct the language in the

problem by demanding how many years *have* elapsed since the age of the father was 4 times that of his son. The equation to the problem may then be obtained from equation (1) by changing x into $-x$.

(83.) Before closing this discussion, we will interpret the symbols $\frac{A}{0}$, $\frac{0}{A}$, $\frac{0}{0}$.

If in equation (2), in the preceding article, we make $a=c$, the value of x becomes $\frac{m-n}{0}$, or in general, $\frac{A}{0}$. Now, the value of a fraction *increases* as the denominator *diminishes*, the numerator remaining the same. Hence, when the denominator is *very small*, the value of the fraction must be *very great*, and when the denominator is *infinitely small*, or *nothing*, the value of the fraction is *infinitely great*, or *infinity*. The symbol for infinity is ∞. Therefore, $\frac{A}{0}=\infty$.

If in equation (2), we make $m=n$, the value of x becomes $\frac{0}{a-c}$, or in general, $\frac{0}{A}$. By a reasoning similar to that which has just been employed, we may show that $\frac{0}{A}=0$.

If in equation (2), we make $m=n$, and at the same time make $a=c$, the value of x becomes $\frac{0}{0}$. In this case, equation (1) becomes $ax+m=ax+m$, and it is plain that the value of x, $\frac{0}{0}$, may be represented by any number whatever. Hence, $\frac{0}{0}$ is the symbol of an *indeterminate quantity.*

We must not, however, always conclude that when the solution of a problem is $\frac{0}{0}$, the problem is indeterminate. Suppose that the solution of a certain problem is,

$$x=\frac{a^3-b^3}{a^2-b^2}$$

If we make $a=b$, this value of x becomes $\frac{0}{0}$. But if we divide the numerator, a^3-b^3, and the denominator, a^2-b^2, by $a-b$, their greatest common measure, the value of x becomes,

$$x=\frac{a^2+ab+b^2}{a+b}$$

If we now make the hypothesis that $a=b$, we have,

$$x=\frac{3a^2}{2a}=\frac{3}{2}a$$

This is the true value of x, when $a=b$. We therefore conclude that when the solution of a problem is a fraction which reduces to $\frac{0}{0}$, when a particular hypothesis is made, we must first determine whether the terms of this fraction have a common measure. If no common measure exists, then the value of x must be indeterminate, that is, x may have any value whatever. But if the terms of the fraction have a common measure, the fraction must be reduced by dividing both terms by this common measure. In the reduced fraction, we must make the same hypothesis, and it will then assume one of the three forms $\frac{A}{B}$, $\frac{A}{0}$, $\frac{0}{0}$.

In the first form, the value of x is *determinate;* in the second, it cannot be expressed in *finite numbers;* in the third, it is *indeterminate.*

CHAPTER V.

INVOLUTION, EVOLUTION, RADICAL QUANTITIES, AND IMAGINARY QUANTITIES.

INVOLUTION.

(84.) INVOLUTION *is the raising of a quantity to any required power.*

The involution of algebraical quantities may be divided into two cases.

CASE I.

To involve a monomial.

(85.) Let it be required to raise $2a^3$ to the third power. By definition 15, the third power of $2a^3=2a^3\times 2a^3\times 2a^3=8a^9$ by the rules of multiplication.

Let it be required to raise $-5c$ to the third power. By definition 15, the third power of $-5c=-5c\times -5c\times -5c=-125c^3$. In this example, there is an *odd* number of *negative* factors, and the product, $-125c^3$, is therefore *negative.*

Let it be required to raise $-2n$ to the fourth power. By definition 15, the fourth power of $-2n=-2n\times -2n\times -2n\times -2n=+16n^4$. In this example, there is an *even* number of *negative* factors, and the product, $16n^4$, is therefore *positive.*

The fourth power of $3x^{\frac{1}{2}}=3x^{\frac{1}{2}}\times 3x^{\frac{1}{2}}\times 3x^{\frac{1}{2}}\times 3x^{\frac{1}{2}}=81x^2$.

The third power of $4x^{\frac{2}{3}}=4x^{\frac{2}{3}}\times 4x^{\frac{2}{3}}\times 4x^{\frac{2}{3}}=64x^2$.

In these examples, let it be observed that the exponents of the literal parts in the required power may be obtained by multiplying the exponents of the letters in the given quantities by the index of the required powers; and that the co-efficients of the literal parts in the required powers may be obtained by raising the numerical co-efficients in the given quantities to a power denoted by the index of the required power. Thus, the fourth power of $3x^{\frac{1}{3}}$ is formed by multiplying the exponent $\frac{1}{3}$ by 4, and raising 3, the co-efficient, to the fourth power. Hence, the fourth power of $3x^{\frac{1}{3}}$ is $81x^{\frac{4}{3}}$. The fourth power of $2a^2b$ is $16a^8b^4$. The exponent of b is 1 understood.

It is often convenient to indicate powers of quantities by employing the exponent of the required power. Thus the third power of $2x^{\frac{1}{4}}$ is expressed $(2x^{\frac{1}{4}})^3$. The fifth power of $4a^2$ is expressed $(4a^2)^5$.

Monomials may be involved by the aid of the following

RULE.

Multiply the exponents of the different letters by the index of the required power, and to this power of the letters prefix that of the co-efficient.

EXAMPLES.

1. What is the fourth power of $5a^2b$?

Ans. $625a^8b^4$.

2. What is the fourth power of $7a^3b^{\frac{2}{3}}$?

Ans. $2401a^{12}b^{\frac{8}{3}}$.

3. What is the fifth power of $3ac^{\frac{2}{5}}$? *Ans.* $243a^5c^2$.

4. What is the sixth power of $2ac^{\frac{1}{3}}$? *Ans.* $64a^6c^2$.

5. What is the nth power of a^2c^3? *Ans.* $a^{2n}c^{3n}$.

6. What is the third power of $-3a^2c^{\frac{1}{3}}$?

Ans. $-27a^6c$.

7. What is the fifth power of $\dfrac{a^2}{c^{\frac{2}{5}}}$?

Ans. $\dfrac{a^{10}}{c^2}$.

8. What is the sixth power of $-3ac^{\frac{2}{3}}$?

Ans. $729a^6c^4$.

9. What is the fourth power of $5a^{\frac{3}{12}}c^2$?

Ans. $625ac^8$.

10. What is the third power of $\dfrac{11a^{\frac{3}{6}}}{2c^{\frac{4}{6}}}$?

Ans. $\dfrac{1331a^{\frac{3}{2}}}{8c^2}$.

11. What is the seventh power of $-2a^{\frac{2}{21}}$?

Ans. $-128a^{\frac{2}{3}}$.

12. What is the second power of $21a^{\frac{1}{2}}c^{\frac{2}{3}}$?

Ans. $441ac^{\frac{4}{3}}$.

CASE II.

To involve a polynomial.

(86.) A polynomial is involved to any required power by actual multiplication. Thus, the third power of $a+b+c$ is found by multiplying it by itself, and that product by $a+b+c$. The quantity to be involved must be used as a factor as many times as there are units in the exponent of the required power.

When a quantity to be involved has an exponent, this exponent may be multiplied by the index of the required power, and the quantity may then be raised to the required power. Thus, to find the second power of $(a+x)^3$, we can multiply the exponent 3 by 2, and then involve $a+x$ to the sixth power. This course will be found necessary when the exponent of the given quantity is a

fraction. For example, let it be required to raise $(a+x)^{\frac{1}{2}}$ to the fourth power. We multiply the exponent by 4, and we obtain 2. Hence, the fourth power of $(a+x)^{\frac{1}{2}}$ is expressed $(a+x)^2=a^2+2ax+x^2$.

BINOMIAL THEOREM FOR POSITIVE AND INTEGRAL EXPONENTS.

(87.) It is obvious that polynomials may be raised to any required power by actual multiplication; but in many cases, this labor would be very long and tedious. As it is often necessary to involve *binomials* in solving problems in pure and quadratic equations, we will show in what way we can write down any power of a binomial, when the exponent is a positive integer, without the labor of multiplication. When a binomial, as $x+a$, is raised to any power, it is found that the successive terms bear a certain relation to each other, and that this relation is the same in all cases. This law, when expressed in algebraical characters, is called the "Binomial Theorem." It was discovered by Sir Isaac Newton, and it is very frequently used in the higher branches of mathematical analysis. For the present, we shall only show the application of the Theorem in the case where the exponent is a positive integer. In a subsequent chapter, we shall give a rigorous demonstration of the Theorem.

For the purpose of examining the powers of a binomial, we will find some of the powers of the binomial, $x+a$, and arrange them in a table.

$$\begin{array}{ll}
x+a & \\
\underline{x+a} & \\
x^2+\ xa & \\
\underline{\qquad xa+a^2} & \\
x^2+2xa+a^2= & \text{2d power.} \\
\underline{x+a} & \\
x^3+2x^2a+\ xa^2 & \\
\underline{\qquad x^2a+2xa^2+a^3} & \\
x^3+3x^2a+3xa^2+a^3. & \text{3d power.}
\end{array}$$

$$
\begin{array}{lr}
x^3+3x^2a+3xa^2+a^3 & \\
x+a & \\
\hline
x^4+3x^3a+3x^2a^2+\ xa^3 & \\
\qquad x^3a+3x^2a^2+3xa^3+a^4 & \\
\hline
x^4+4x^3a+6x^2a^2+4xa^3+a^4= & \text{4th power.} \\
x+a & \\
\hline
x^5+4x^4a+\ 6x^3a^2+\ 4x^2a^3+\ xa^4 & \\
\qquad x^4a+\ 4x^3a^2+\ 6x^2a^3+4xa^4+a^5 & \\
\hline
x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5= & \text{5th power.}
\end{array}
$$

TABLE OF THE POWERS OF $x+a$.

$x+a$	$x+a$
$(x+a)^2$	$x^2+2xa+a^2$
$(x+a)^3$	$x^3+3x^2a+3xa^2+a^3$
$(x+a)^4$	$x^4+4x^3a+6x^2a^2+4xa^3+a^4$
$(x+a)^5$	$x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$
$(x+a)^6$	$x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6$
$(x+a)^7$	$x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+15x^2a^5+7xa^6+a^7$

In this table, the quantities in the right hand column are called the *developments* or *expansions* of $x+a$ raised to the first, second, third, &c., power, and the corresponding quantities in the left hand columns are the expressions for these powers.

By examining these expansions, we may make the following observations:

I. In the first term of the expansion, the first term of the binomial is found raised to the given power, and the exponents of x decrease by unity in each successive term of the expansion. The first term of the binomial is found in each term of the expansion except the last.

II. The second term of the binomial is found in each term of the expansion except the first, and its exponents increase by unity in each successive term. In the last term of the expansion, the exponent of a is the same as that of the given power.

III. The co-efficient of the first term in each expansion is unity and the co-efficient of the second term in any expansion is the exponent of the power to which the quantity is to be raised. Thus, in the fourth power of $x+a$, the co-efficient of the second term is 4, in the sixth power, 6 ; and so on.

IV. The co-efficient of any term may be found by multiplying the coefficient of the preceding term by the exponent of x in that term, and dividing the product by the exponent of a increased by unity in that term. Thus, to find the co-efficient of the fifth term in the expansion of the sixth power of $x+a$, we multiply 20, the co-efficient of the preceding term, by 3, the exponent of x in that term, and divide the product, 60, by 4, the exponent of a in that term increased by unity, and obtain 15 for the co-efficient of the fifth term.

V. All the signs of the terms in the expansion of the successive powers of $x+a$ are positive. The development of any power of $x-a$ is the same as the development of the same power of $x+a$, except that the signs are alternately positive and negative. Thus, $(x-a)^3=x^3-3x^2a+3x^2a^2-a^3$.

EXAMPLES.

1. What is the third power of $3a+2c$?

In this example; let $m=3a$, $n=2c$; then $3a+2c=m+n$, $(3a+2c)^3=(m+n)^3=m^3+3m^2n+3mn^2+n^3$.

$$\begin{array}{lll} \text{Since,} & m=\ 3a & (1) \\ \text{And} & n=\ 2c & (2) \\ \text{We have,} & \left\{\begin{array}{l} m^3=27a^3 \\ m^2=\ 9a^2 \\ n^3=\ 8c^3 \\ n^2=\ 4c^2 \end{array}\right. & \begin{array}{l} (3)=(1)^3 \\ (4)=(1)^2 \\ (5)=(2)^3 \\ (6)=(2)^2 \end{array} \end{array}$$

Substituting these values of m^3, m^2, n^3, n^2, in the expansion of $m+n$, we find that,

$(m+n)^3=(3a+2c)^3=27a^3+3\times 9a^2\times 2c+3\times 3a\times 4c^2+8c^3$
$=27a^3+54a^2c+36ac^2+8c^3$.

2. What is the fourth power of $a+c$?

Ans. $a^4+4a^3c+6a^2c^2+4ac^3+c^4$.

3. What is the square of $4ax+x+1$?

Ans. $16a^2x^2+8ax^2+8ax+x^2+2x+1$.

4. What is the sixth power of $(a+x)^{\frac{1}{2}}$?

Ans. $a^3+3a^2x+3ax^2+x^3$.

5. What is the square of $a+b+c$?

Ans. $a^2+2ab+2ac+b^2+2bc+c^2$.

6. What is the cube of $a-2c$?

Ans. $a^3-6a^2c+12ac^2-8c^3$.

7. What is the sum of the cubes of $a+c$, and $a-c$?

Ans. $2a^3+6ac^2$.

8. What is the sum of the fourth powers of $x+y$, and $x-y$?

Ans. $2x^4+12x^2y^2+2y^4$.

9. What is the sum of the fifth powers of $a+c$, and $(a-c)$?

Ans. $2a^5+20a^3c^2+10ac^4$.

10. What is the sixth power of $(a+c)^{\frac{1}{3}}$?

Ans. $(a+c)^2=a^2+2ac+c^2$.

11. What is the second power of $9x+\frac{1}{x}$?

Ans. $81x^2+18+\frac{1}{x^2}$.

12. What is the second power of $4+x^{\frac{1}{2}}$?

Ans. $16+8x^{\frac{1}{2}}+x$.

13. What is the square of $x-\frac{(x^4-a^4)^{\frac{1}{2}}}{a}$?

Ans. $x^2-\frac{2x}{a}(x^4-a^4)^{\frac{1}{2}}+\frac{x^4-a^4}{a^2}$.

14. What is the square of $y^{\frac{1}{4}}+3x^{\frac{1}{2}}$?

Ans. $y^{\frac{1}{2}}+6y^{\frac{1}{4}}x^{\frac{1}{2}}+9x$.

15. What is the cube of $a^{\frac{1}{3}}+c^{\frac{1}{3}}$?

Ans. $a+3a^{\frac{2}{3}}c^{\frac{1}{3}}+3a^{\frac{1}{3}}c^{\frac{2}{3}}+c$.

16. What is the fourth power of $a^{\frac{1}{2}}+c^{\frac{1}{2}}$?

Ans. $a^2+4a^{\frac{3}{2}}c^{\frac{1}{2}}+6ac+4a^{\frac{1}{2}}c^{\frac{3}{2}}+c^2$.

EVOLUTION.

(88.) Evolution is the extracting of roots. It is the reverse of Involution.

The square root of any quantity may be denoted by the index $\frac{1}{2}$. For, the square root of any quantity, as a, is a quantity which multiplied by itself will produce the given quantity. Now, $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$, by the principle of multiplication. Hence, the square root of a may be written $a^{\frac{1}{2}}$. In general, the nth root of a is $a^{\frac{1}{n}}$, since the product of n factors, each of which is $a^{\frac{1}{n}}$, is a. We see, therefore, that roots, as well as powers, may be correctly represented by exponents. On this principle we may denote a root of any quantity, by enclosing this quantity within the parenthesis, and then regarding it as a single quantity. Thus, the square root of $a^2+2ac+c^2$ may be written, $(a^2+2ac+c^2)^{\frac{1}{2}}$.

Evolution may be divided into three cases.

CASE I.

To extract any root of a monomial.

RULE.

Extract the root of the numeral co-efficient; and then extract the root of the literal part by dividing the exponent of each letter by the denominator of the fraction, which is the index of the required root.

When the root of a quantity cannot be accurately found, we may indicate it by a fractional exponent. Thus, the cube root of $4a^2$ may be indicated by enclosing it in a parenthesis, and then treating it as a single quantity. It is written $(4a^2)^{\frac{1}{3}}$.

(89.) Since all *even* powers of a negative quantity are positive, it follows that all even roots of a positive quantity may be either positive or negative. All odd roots of a negative quantity are negative. An even root of a negative quantity is *impossible.* For example, we cannot take the square root of $-a^2$, since no quantity multiplied by *itself* will produce $-a^2$.

EXAMPLES.

1. What is the cube root of $64a^6b^3$?

In this example, the cube root of the co-efficient is 4, and dividing the exponents of the letters by 3, the denominator of the fractional index which denotes the cube root, we have a^2b for the cube root of the literal part. Hence the root is $4a^2b$.

2. What is the cube root of $27a^3c^9$? *Ans.* $3ac^3$.

3. What is the square root of $625a^{12}c$?

Ans. $25a^6c^{\frac{1}{2}}$.

4. What is the seventh root of $128a^{14}x^3$?

Ans. $2a^2x^{\frac{3}{7}}$.

5. What is the fourth root of $\frac{16}{81}\sqrt[3]{c^2}$? *Ans.* $\frac{2}{3}c^{\frac{1}{6}}$.

6. What is the square root of $64a^{\frac{1}{4}}$? *Ans.* $8a^{\frac{1}{8}}$.

7. What is the fourth root of $16a^6\sqrt{c}$?

Ans. $2a^{\frac{3}{2}}c^{\frac{1}{8}}$.

8. What is the fifth root of $32a^{15}c^{\frac{1}{2}}$? *Ans.* $2a^3c^{\frac{1}{10}}$.

9. What is the sixth root of $729a^{12}c^3$?

Ans. $3a^2\sqrt{c}$.

10. What is the square root of $49m^{\frac{1}{2}}n^{\frac{1}{3}}$?

Ans. $7m^{\frac{1}{4}}n^{\frac{1}{6}}$.

11. What is the cube root of $\frac{1}{27}a^3b^{\frac{1}{2}}$? *Ans.* $\frac{1}{3}ab^{\frac{1}{6}}$.

12. What is the square root of $100a^4x^{\frac{1}{5}}$?

Ans. $10a^2x^{\frac{1}{10}}$.

CASE II.

To extract the root of a polynomial.

We shall only show in this place how to extract the square and cube roots of polynomials.

To extract the square root of a polynomial.

(90.) Since the square of $a+b$ is $a^2+2ab+b^2$, it follows that $a+b$ is the square root of $a^2+2ab+b^2$. Now the square root of the first term a^2, is a, the first term of the root. If we subtract a^2, the square of the first term of the root, from $a^2+2ab+b^2$, we have for a remainder, $2ab+b^2=(2a+b)\times b$. This remainder is composed of two factors, one of which, b, the second term of the root, may be obtained by dividing the first term of the remainder, $2ab$, by twice the first term of the root. If we add b to $2a$ we obtain $2a+b$, the complete divisor of $2ab+b^2$. This complete divisor, multiplied by b, the second term in the root, and the product subtracted from the remainder, finishes the operation.

$$\left\{\begin{array}{rl} & \qquad\qquad\qquad\quad \text{Root.} \\ & a^2+2ab+b^2(a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline & \qquad 0 \end{array}\right.$$

If the root has three terms, it may be represented by $a+b+c$, and its square is $(a+b)^2+2c(a+b)+c^2$. Now, we can find the root of the first part, $(a+b)^2=a^2+2ab+b^2$, as above, and then find c from $2c(a+b)$ in the same way that b was derived from $2ab$, in the last example. It may be shown that, in a similar manner, the root of a polynomial may be found, which has four or a greater number of terms.

Hence, the square root of a polynomial may be found by the following

RULE.

I. *Arrange the polynomial according to the powers of one of its letters, as in division.*

II. *Extract the square root of the first term, and subtract its square from the polynomial. Bring down the next two terms of the polynomial for a dividend, and divide the first term of the dividend by twice the first term of the root just found. The quotient will be the second term af the root. Add the second term of the root to twice the first term of the root, and multiply the sum by the second term of the root, and subtract the product from the dividend. To the remainder add two or more terms of the polynomial, and proceed as before.*

EXAMPLES.

1. What is the square root of $a^4+4a^3c+6a^2b^2+4ac^3+c^4$?

Operation.

$$
\begin{array}{rl}
 & \text{Root.} \\
 & a^4+4a^3c+6a^2c^2+4ac^3+c^4\,(a^2+2ac+c^2 \\
 & a^4 \\
2a^2+2ac) & 4a^3c+6a^2c^2 \\
 & 4a^3c+4a^2c^2 \\
2a^2+4ac+c^2) & 2a^2c^2+4ac^3+c^4 \\
 & 2a^2c^2+4ac^3+c^4 \\
 & 0
\end{array}
$$

2. What is the square root of $16a^2x^2+8ax^2+8ax+x^2+2x+1$?

Operation.

$$
\begin{array}{rl}
 & \text{Root.} \\
 & 16a^2x^2+8ax^2+8ax+x^2+2x+1\,(4ax+x+1 \\
 & 16a^2x^2 \\
8ax+x) & 8ax^2+8ax+x^2 \\
 & 8ax^2 \qquad x^2 \\
8ax+2x+1) & 8ax \qquad +2x+1 \\
 & 8ax \qquad +2x+1 \\
 & 0
\end{array}
$$

3. What is the square root of $x^4-ax^3+\frac{1}{4}a^2x^2$?

Ans. $x^2-\frac{1}{2}ax$.

4. What is the square root of $9x^2-30ax-3a^2x+25a^2+5a^3+\dfrac{a^4}{4}$?

Ans. $3x-5a-\dfrac{a^2}{2}$.

5. What is the square root of $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$?

Ans. $2x^2+2ax+4b^2$.

6. What is the square root of $49x^4-28x^3-17x^2+6x+\frac{9}{4}$?

Ans. $7x^2-2x-\frac{3}{2}$.

7. What is the square root of $4x^4-16x^3+24x^2-16x+4$?

Ans. $2x^2-4x+2$.

8. What is the square root of $16x^4+24x^3+89x^2+60x+100$?

Ans. $4x^2+3x+10$.

9. What is the square root of $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$?

Ans. $3x^3-2x^2+x-4$.

10. What is the square root of $4a^4-4a^3+5a^2-2a+1$?

Ans. $2a^2-a+1$.

11. What is the square root of $4a^2-4am-4an+m^2+2mn+n^2$?

Ans. $2a-m-n$.

12. What is the square root of $4a^2+4an+n^2-4a-2n+1$?

Ans. $2a+n-1$.

13. What is the square root of $9a^2-24ac+6a+16c^2-8c+1$?

Ans. $3a-4c+1$.

14. What is the square root of $4x^4+4ax^2+4x^2+a^2+2a+1$?

Ans. $2x+a+1$.

To extract the cube root of a polynomial.

(91.) Since the *cube* of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, it follows that the *cube root* of $a^3+3a^2b+3ab^2+b^3$ is $a+b$. Now, the first term of this root, is obtained by extracting the cube root of the first term of the polynomial, $a^3+3a^2b+3ab^2+b^3$. Subtract the cube of the first term of the root from the polynomial, and the remainder is $3a^2b+3ab^2+b^3$. By resolving this remainder into its factors, we find that, $3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)\times b$. If, then, we divide the remainder by $3a^2+3ab+b^2$, we shall obtain the second term of the root. We may call $3a^2+3ab+b^2$ the *true* or *complete divisor* of the remainder. We may also observe that the second term of the root may be obtained by dividing the first term of the remainder, $3a^2b$, by three times the square of the first term of the root, that is, by $3a^2$. We will call $3a^2$ the *trial divisor*, since it may be used for determining the next term in the root.

For convenience in calculation, the *trial* and *complete divisors* may be formed in the following manner:

Write the first term of the root in a column, designated Column I., and its square in a column designated Column II. Add the first term of the root to the term in Column I., multiply the sum by the first term in the root, and add the product to the term in the second column. The sum thus obtained is the *trial divisor*. The second term in the root is obtained by dividing the first term of the remainder by the *trial divisor*. Now add the first term in the root to the last term in Column I., and to this sum add the second term in the root, and the latter sum is the next term in Column I., and if this term be multiplied by the second term in the root, and the product be added to the trial divisor, the sum will be the *true* or *complete divisor*. Multiply the *true divisor* by the second term in .the root, and subtract the product from the first remainder, and the work is finished. We have, then, for extracting the cube root of $a^3+3a^2b+3ab^2+b^3$, the following

Operation.

Col. I.	Col. II.	
		$a^3+3a^2b+3ab^2+b^3 \mid \underline{a+b}$
a	a^2	a^3
a	$2a^2$	$3a^2b+3ab^2+b^3$=Remainder.
$2a$	$3a^2$=Trial divisor.	$3a^2b+3ab^2+b^3$
a	$3ab+b^2$	0
$3a$	$3a^2+3ab+b^2$=True div.	
$+b$		
$3a+b$		

If the root has three terms, it may be represented by $a+b+c$, and its cube may be written $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$ $=a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3$. Now, we may return to the root of this polynomial by finding the first part of the root, $a+b$, as before, and then deriving c by means of $a+b$, in the same way that b was derived by means of a. In a similar manner, we may find the cube root of a polynomial, when its root has four, or any number of terms.

From what has been said, we conclude that the cube root of any polynomial may be found by the following

RULE.

I. *Arrange the terms of the polynomial with reference to the powers of one of its letters.*

II. *Extract the cube root of the first, or left hand term, for the first term of the required root. Write the first term of the root in a column, designated Column* I., *and its square in a column, designated Column* II. *Multiply the term in Column* II. *by the first term in the root, and subtract the product from the given polynomial, and take three terms of the remainder for the* FIRST DIVIDEND.

III. *Add the first term in the root to the term in Column* I., *multiply the sum by the first term in the root, and add the product to the term in Column* II. *The sum thus obtained is the* FIRST TRIAL DIVISOR. *Add the first term in the root to the last term in Column* I., *and the sum is the next term in this column.*

IV. *Divide the first term of the first dividend by the trial divisor, and the quotient is the next, or second term of the root. Add the second term of the root to the last term in Column* I., *for the next term in this column, and then multiply this term in Column* I. *by the second term in the root, and add the product to the first trial divisor. The sum thus obtained is the* FIRST TRUE DIVISOR. *Multiply the true divisor by the second term in the root, and subtract the product from the first dividend. To the remainder annex the next three terms of the polynomial for the* SECOND DIVIDEND. *To the last term in Column* I., *add the last term in the root for the next term in Column* I., *and multiply this term of Column* I. *by the last term in the root, and add the product to the first true divisor. The sum thus obtained is the* SECOND TRIAL DIVISOR. *Then, with the second trial divisor, and the second dividend, proceed as before.**

EXAMPLES.

1. What is the cube root of $8a^3+12a^2c+6ac^2+c^3$?

Solution.

Col. I.	Col. II.		Root.
		$8a^3+12a^2c+6ac^2+c^3$	$2a+c$
$2a$	$4a^2$	$8a^3$	
$2a$	$8a^2$	$12a^2c+6ac^2+c^3$	
$4a$	$12a^2$ = First trial divisor.	$12a^2c+6ac^2+c^3$	
$2a$	$6ac+c^2$	0	
$6a$	$12a^2+6ac+c^2$ = First true div.		
$+c$			
$6a+c$			

If, in obtaining the several terms in Column I. and Column II., we perform the additions mentally, these columns will assume a more compact form, as represented in the following operations.

* NOTE.—From the rule here given for extracting the cube root of a polynomial, we might easily deduce a rule for extracting the cube root of any number. See Hutton's Mathematics, late edition, and Perkins' Higher Arithmetic.

2. What is the cube root of $27c^6-54c^5+63c^4-44c^3+21c^2-6c+1$?

Operation.

Col. I.	Col. II.	
$3c^2$	$9c^4$	$27c^6-54c^5+63c^4-44c^3+21c^2-6c+1(3c^2-2c+1$
$6c^2$	$27c^4$	$27c^6$
$9c^2$	$27c^4-18c^3+4c^2$	$-54c^5+63c^4-44c^3$
$9c^2-2c$	$27c^4-36c^3+12c^2$	$-54c^5+36c^4-8c^3$
$9c^2-4c$	$27c^4-36c^3+21c^2-6c+1$	$27c^4-36c^3+21c^2-6c+1$
$9c^2-6c$		$27c^4-36c^3+21c^2-6c+1$
$9c^2-6c+1$		0

3. What is the cube root of $c^6+6c^5-40c^3+96c-64$?

Col. I.	Col. II.	
c^2	c^4	$c^6+6c^5-40c^3+96c-64(c^2+2c-4$
$2c^2$	$3c^4$	c^6
$3c^2$	$3c^4+6c^3+4c^2$	$6c^5-40c^3+96c$
$3c^2+2c$	$3c^4+12c^3+12c^2$	$6c^5+12c^4+8c^3$
$3c^2+4c$	$3c^4+12c^3-24c+16$	$-12c^4-48c^3+96c-64$
$3c^2+6c$		$-12c^4-48c^3+96c-64$
$3c^2+6c-4$		0

4. What is the cube root of $8a^6-12a^5+18a^4-13a^3+9a^2-3a+1$? *Ans.* $2a^2-a+1$.

5. What is the cube root of $27c^3+27c^2+9c+1$? *Ans.* $3c+1$.

6. What is the cube root of $8x^3+36x^2+54x+27$? *Ans.* $2x+3$.

7. What is the cube root of $x^6+6x^5-40x^3+96x-64$? *Ans.* x^2+2x-4.

8. What is the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$? *Ans.* x^2-2x+1.

9. What is the cube root of $a^3+6a^2c+12ac^2+8c^3$? *Ans.* $a+2c$.

10. What is the cube root of $8c^3-36c^2+54c-27$? *Ans.* $2c-3$.

11. What is the cube root of $x^3+6x^2+12x+8$? *Ans.* $x+2$.

12. What is the cube root of $8a^3-84a^2x+294ax^2-343x^3$? *Ans.* $2a-7x$.

13. What is the cube root of $x^6-6cx^5+12c^2x^4-8c^3x^3$? *Ans.* x^2-2cx.

RADICAL QUANTITIES.

(92.) A Radical quantity is one that is affected with a radical sign or fractional exponent, without which it cannot be accurately expressed. Thus, the expressions, $\sqrt{3}$, $\sqrt[3]{2a}$, $(2a+c^2)^{\frac{1}{3}}$ $n^{\frac{1}{4}}$, $c^{\frac{3}{5}}$, are radical quantities. They are also called *surds* and *irrational quantities.* The square root of a quantity is a radical of the second degree, the cube root of a quantity is a radical of the third degree, and, in general, the nth root of a quantity is a radical of the nth degree.

(93.) Radical quantities are similar when each is affected with the same radical sign, and the quantity under the radical is the same in each. Thus, $5a\sqrt{2b}$, $3c\sqrt{2b}$, are *similar radical quantities.* The expressions, $(3a+c)^{\frac{1}{4}}$, $4c(3a+c)^{\frac{1}{3}}$ are *dissimilar radicals.*

REDUCTION OF RADICALS.

(94.) The reduction of radicals consists in changing their form without altering their value. There are three cases.

CASE I.

To reduce a rational quantity to the form of a radical.

RULE.

Raise the quantity to a power denoted by the degree of the radical, and then affect it with the sign of the required radical.

EXAMPLES.

1. Reduce $7c$ to the form of cube root.

In this example, the radical is of the third degree. Hence, we must raise $7c$ to the third power, and affect this power with the sign of the cube root. $(7a)^3=243a^3$. By extracting the cube root we have $7a=(243a^3)^{\frac{1}{3}}$; or it may be written, $7a=\sqrt[3]{243a^3}$.

2. Reduce $3a$ to the form of the fourth root.

Ans. $(81a^4)^{\frac{1}{4}}$.

3. Reduce $5a^{\frac{1}{2}}$ to the form of the square root.

Ans. $\sqrt{25a}$.

4. Reduce $4c^{\frac{2}{3}}$ to the form of the cube root.

Ans. $(64c^2)^{\frac{1}{3}}$.

5. Reduce $45c^{\frac{1}{6}}$ to the form of the cube root.

Ans. $\left(91125c^{\frac{1}{2}}\right)^{\frac{1}{3}}$.

6. Reduce $4c^2$ to the form of the nth root.

Ans. $(4^n c^{2n})^{\frac{1}{n}}$.

7. Reduce $16c^3$ to the form of a radical whose exponent is $\frac{4}{3}$.

Ans. $\left(16^{\frac{3}{4}}c^{\frac{9}{4}}\right)^{\frac{4}{3}} = \left(8c^{\frac{9}{4}}\right)^{\frac{4}{3}}$.

8. Reduce $125c^{\frac{2}{4}}$ to the form of a radical whose exponent is $\frac{3}{2}$.

Ans. $\left(125^{\frac{2}{3}}c^{\frac{1}{3}}\right)^{\frac{3}{2}} = \left(25c^{\frac{1}{3}}\right)^{\frac{3}{2}}$.

9. Reduce $32ab^{\frac{10}{9}}$ to the form of a radical whose exponent is $\frac{5}{3}$.

Ans. $\left(32^{\frac{3}{5}}a^{\frac{3}{5}}b^{\frac{30}{45}}\right)^{\frac{5}{3}} = \left(8a^{\frac{3}{5}}b^{\frac{2}{3}}\right)^{\frac{5}{3}}$.

10. Reduce $64a^{\frac{12}{5}}$ to the form of a radical whose exponent is $\frac{6}{5}$.

Ans. $(32a^2)^{\frac{6}{5}}$.

11. Reduce $a^{\frac{2}{5}}x^2$ to the form of the nth root.

Ans. $\left(a^{\frac{2n}{5}}x^{2n}\right)^{\frac{1}{n}}$.

12. Reduce $8cx^{\frac{1}{2}}$ to the form of a radical whose exponent is $\frac{1}{3}$.

Ans. $\left(512c^3x^{\frac{3}{2}}\right)^{\frac{1}{3}}$.

CASE II.

(95.) *To reduce radicals which have different exponents to equivalent ones having a common index.*

RULE.

Reduce the exponents to fractions having a common denominator, and then raise each quantity to a power denoted by the numerator of its exponent, and the reciprocal of the common denominator will be the common index of each.

This rule depends upon the obvious principle, that the values of the exponents are not altered, and therefore the value of the radicals themselves are not changed.

6*

EXAMPLES.

1. Reduce $2^{\frac{1}{3}}$ and $4^{\frac{1}{4}}$ to radicals having a common index. In this example, the exponents reduced to fractions having a common denominator become $\frac{4}{12}$ and $\frac{3}{12}$. Hence, $2^{\frac{1}{3}}=2^{\frac{4}{12}}=16^{\frac{1}{12}}$; and $4^{\frac{1}{4}}=4^{\frac{3}{12}}=64^{\frac{1}{12}}$.

2. Reduce $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ to radicals having a common index.

Ans. $8^{\frac{1}{6}}$, and $9^{\frac{1}{6}}$.

3. Reduce $2^{\frac{2}{3}}$ and $3^{\frac{1}{4}}$ to radicals having a common index.

Ans. $(256)^{\frac{1}{12}}$, and $(27)^{\frac{1}{12}}$.

4. Reduce $3^{\frac{1}{8}}$ and $2^{\frac{1}{4}}$ to radicals having a common index.

Ans. $3^{\frac{1}{8}}$, and $4^{\frac{1}{8}}$.

5. Reduce $6^{\frac{1}{3}}$ and $4^{\frac{1}{2}}$ to radicals having a common index.

Ans. $(36)^{\frac{1}{6}}$, and $(64)^{\frac{1}{6}}$.

6. Reduce $3(3)^{\frac{1}{4}}$ and $2^{\frac{1}{12}}$ to radicals having a common index.

Ans. $3(27)^{\frac{1}{12}}$, and $2^{\frac{1}{12}}$.

7. Reduce $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$ to radicals having a common index.

Ans. $(a^n)^{\frac{1}{mn}}$, and $(b^m)^{\frac{1}{mn}}$.

8. Reduce $m^{\frac{2}{9}}$ and $n^{\frac{1}{6}}$ to radicals having a common index.

Ans. $(m^4)^{\frac{1}{18}}$, and $(n^3)^{\frac{1}{18}}$

9. Reduce $4^{\frac{1}{5}}$ and $3^{\frac{1}{3}}$ to radicals having a common index.

Ans. $(4^3)^{\frac{1}{15}}$, and $(3^5)^{\frac{1}{15}}$.

10. Reduce $a+c$ and $(a-c)^{\frac{1}{2}}$ to radicals having a common index.

Ans. $(a^2+2ac+c^2)^{\frac{1}{2}}$, and $(a-c)^{\frac{1}{2}}$.

11. Reduce $(a+c)^{\frac{1}{3}}$ and $(a-c)^{\frac{1}{2}}$ to radicals having a common index.

Ans. $[(a+c)^2]^{\frac{1}{6}}$, and $[(a-c)^3]^{\frac{1}{6}}$.

12. Reduce $2^{\frac{1}{2}}$ and $3^{\frac{1}{5}}$ to radicals having a common index.

Ans. $(32)^{\frac{1}{10}}$, and $(9)^{\frac{1}{10}}$.

CASE III.

(96.) *To reduce radical quantities to their simplest forms.*

Before giving the rule for this case, we will establish the following proposition on which it depends.

PROPOSITION.

The product of the nth roots of any number of quantities is equal to the nth root of their product.

Take two quantities, a and b, and the demonstration will be the same for any number. Then we are to prove that, $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$; or, what is the same, $a^{\frac{1}{n}}\times b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}$.

Let P represent the product of their nth roots.

$$\text{Then } P=\sqrt[n]{a}\times\sqrt[n]{b}. \qquad (1)$$

By raising each member to the nth power, we have

$$P^n=a\times b=ab \qquad (2)$$

By extracting the nth root of equation (2), we have

$$P=\sqrt[n]{ab} \qquad (3)$$

By equating right-hand members of equations (1) and (3), we have $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$; hence, the proposition is established. We will now give the rule for reducing radicals to their simplest forms.

RULE.

Resolve the radical into two such factors that the root of one of them may be accurately taken; then find this root, and multiply it by the other factor.

When the radical cannot be resolved into two such factors, it cannot be made to assume a more simple form.

EXAMPLES.

1. Reduce $\sqrt{147a^2x^3}$ to its simplest form.

We can resolve $147a^2x^3$ into the two factors, $49a^2x^2 \times 3x$, one of which is a perfect square; therefore, $\sqrt{147a^2x^3} = \sqrt{49a^2x^2} \times \sqrt{3x} = 7ax\sqrt{3x}$, since the square root $49a^2x^2$ is $7ax$.

2. Reduce $\sqrt{108a^2c}$ to its simplest form.

Ans. $6a\sqrt{3c}$.

3. Reduce $\sqrt{192a^4c^3}$ to its simplest form.

Ans. $8a^2c\sqrt{3c}$.

4. Reduce $\sqrt{48a^2x}$ to its simplest form.

Ans. $4a\sqrt{3x}$.

5. Reduce $\sqrt{a^2-a^3x}$ to its simplest form.

Ans. $a\sqrt{1-ax}$.

6. Reduce $\sqrt{243c^2n^3}$ to its simplest form.

Ans. $9cn\sqrt{3n}$.

7. Reduce $\sqrt[3]{64a^5c^8}$ to its simplest form.

Ans. $4ac^2\sqrt[3]{a^2c^2}$.

8. Reduce $\sqrt{54a^2+9a^2c}$ to its simplest form.

Ans. $3a\sqrt{6+c}$.

9. Reduce $\sqrt[5]{64\ a^7+32x^6}$ to its simplest form.

Ans. $2\sqrt[5]{2a^7+x^6}$.

10. Reduce $\sqrt{\frac{2}{3}}$ to its simplest form.*

Ans. $\frac{1}{3}\sqrt{6}$.

* NOTE.—When the radical is a fraction, multiply both terms of the fraction by some quantity that will render the denominator such a power that the required root may be taken. Thus, in this example,

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{2\times3}{3\times3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9}} \times \sqrt{6} = \tfrac{1}{3}\sqrt{6}.$$

11. Reduce $\sqrt{\frac{5}{12}}$ to its simplest form.

Ans. $\frac{1}{6}\sqrt{15}$.

12. Reduce $\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}$ to its simplest form.

Ans. $\frac{1}{a+x}\left(\frac{a^2-x^2}{1}\right)^{\frac{1}{2}}$.

ADDITION AND SUBTRACTION OF RADICALS.

(97.) The addition and subtraction of radical quantities may be performed by the following

RULE.

Reduce the radicals to their simplest form; then, if the radicals are similar, add or subtract their rational parts, and to the sum or difference annex the common radical part; but if the radicals are dissimilar after the reduction is made, they can only be added or subtracted, by writing them one after another, with their proper signs.

EXAMPLES.

1. What is the sum of $\sqrt{147a^2c}$ and $\sqrt{48a^2c}$?

By reducing, $\sqrt{147a^2c}=\sqrt{49a^2}\times\sqrt{3c}=7a\sqrt{3c}$

And, $\sqrt{48a^2c}=\sqrt{16a^2}\times\sqrt{3c}=4a\sqrt{3c}$

$\therefore$ their sum $=11a\sqrt{3c}$.

2. What is the sum of $\sqrt{27a^2c}$ and $\sqrt{12a^2c}$?

Ans. $5a\sqrt{3c}$.

3. What is the sum of $\sqrt{243ab^2}$ and $\sqrt{192ab^2}$?

Ans. $17b\sqrt{3a}$.

4. What is the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$?

Ans. $8\sqrt[3]{4}$.

5. What is the difference of $\sqrt{a^2x}$ and $\sqrt{b^2x}$?

Ans. $(a-b)\sqrt{x}$.

6. What is the difference of $5\sqrt{20}$ and $3\sqrt{45}$?

Ans. $\sqrt{5}$.

7. What is the difference of $\sqrt{108ax^2}$ and $\sqrt{48ax^2}$?

Ans. $2x\sqrt{3a}$.

8. What is the difference of $m^{\frac{3}{4}}\sqrt{1+\left(\frac{n}{m}\right)^{\frac{1}{2}}}$ and $n^{\frac{3}{4}}\sqrt{1+\left(\frac{m}{n}\right)^{\frac{1}{2}}}$?

Ans. $\left(m^{\frac{1}{2}}-n^{\frac{1}{2}}\right)\times\sqrt{m^{\frac{1}{2}}+n^{\frac{1}{2}}}$.

9. What is the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$?

Ans. $\frac{4}{5}\sqrt{10}$.

10. What is the sum of $3\sqrt[3]{32}$ and $2\sqrt[3]{54}$?

Ans. $6\sqrt[3]{4}+6\sqrt[3]{2}$.

11. What is the sum of $\sqrt{27}$ and $3\sqrt{75}$?

Ans. $18\sqrt{3}$.

12. What is the sum of $\sqrt{2ax^2-4ax+2a}$ and $\sqrt{2ax^2+4ax+2a}$?

Ans. $2x\sqrt{2a}$.

13. What is the sum of $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{1}{3}}$?

Ans. $\frac{5}{12}\sqrt{12}$.

14. What is the sum of $a\sqrt{1+\left(\frac{b}{a}\right)^{\frac{2}{3}}}$ and $b\sqrt{1+\left(\frac{a}{b}\right)^{\frac{2}{3}}}$?

Ans. $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.*

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* Note.—This is the length of "the longest straight pole that can be put up a chimney, when the height from the floor to the mantel$=a$, and the depth from the front to the back$=b$." See Hutton's Mathematics, Dif. Cal. page 726.

MULTIPLICATION OF RADICALS.

(98.) The multiplication of radical quantities may be performed by the following rule, which is founded on the proposition that was demonstrated in Art. 96.

RULE.

Reduce the radicals so that they shall be of the same degree (Art. 95) ; *and then annex the product of the rational parts to that of the radical parts.*

EXAMPLES.

1. Multiply $\sqrt{8}$ by $\sqrt[3]{48}$.

Here, $\sqrt{8}=8^{\frac{1}{2}}=8^{\frac{3}{6}}=(8^3)^{\frac{1}{6}}=(512)^{\frac{1}{6}}=(64\times 8)^{\frac{1}{6}}=2(8)^{\frac{1}{6}}$.

And, $\sqrt[3]{48}=(48)^{\frac{1}{3}}=(48)^{\frac{2}{6}}=(48^2)^{\frac{1}{6}}=(48\times 48)^{\frac{1}{6}}=(64\times 36)^{\frac{1}{6}}$ $=2(36)^{\frac{1}{6}}$.

Therefore, $\sqrt{8}\times\sqrt[3]{48}=2(8)^{\frac{1}{6}}\times 2(36)^{\frac{1}{6}}=4\sqrt[6]{288}$.

2. Multiply $\sqrt{18}$ by $\sqrt{48}$. *Ans.* $12\sqrt{6}$.

3. Multiply $\sqrt{24a^2x}$ by $\sqrt{12x}$. *Ans.* $12a\sqrt{2}$.

4. Multiply $a^{\frac{1}{3}}$ by $b^{\frac{1}{2}}$. *Ans.* $\sqrt[6]{a^2b^3}$.

5. Multiply $3\sqrt[3]{4}$ by $4\sqrt{3}$. *Ans.* $12\sqrt[6]{432}$.

6. Multiply $a^{\frac{1}{m}}$ by $b^{\frac{1}{n}}$. *Ans.* $(a^nb^m)^{\frac{1}{mn}}$.

7. Multiply $2\sqrt[6]{3}$ by $\sqrt[6]{72}$. *Ans.* $2\sqrt{6}$.

8. Multiply $(a+\sqrt{c})^{\frac{1}{n}}$ by $(a-\sqrt{c})^{\frac{1}{n}}$. *Ans.* $(a^2-c)^{\frac{1}{n}}$.

9. Multiply $4a^{\frac{1}{2}}$ by $13a^{\frac{1}{3}}$. *Ans.* $52\sqrt[6]{a^5}$.

10. Multiply the cube root of 3 by the square root of 2.
Ans. $\sqrt[6]{72}$.

11. Multiply $\sqrt{a}-\sqrt{a-x}$ by $\sqrt{a}+\sqrt{a-x}$.

Ans. x.

12. Multiply $(a+b)^{\frac{1}{2}}$ by $(a+b)^{\frac{1}{3}}$. *Ans.* $\sqrt[6]{(a+b)^5}$.

DIVISION OF RADICALS.

(99.) Before giving the rule for the division of radicals, it will be necessary to establish the following proposition, on which the rule depends.

PROPOSITION.

The quotient of the nth *roots of two quantities is equal to the* nth *root of their quotient.*

Let a and b represent the two quantities; then will $\sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{a\div b}$. Let P=their quotient,

$$\text{Then,}\quad P=\frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad (1)$$

By raising each member to the nth power, we have

$$P^n=\frac{a}{b} \qquad (2)$$

By extracting the nth root of each member of equation (2), we have

$P=\sqrt[n]{\frac{a}{b}}$; hence the proposition is established.

RULE.

Reduce the radicals so that they shall be of the same degree; and then annex the quotient of the rational parts to the quotient of the radical parts.

EXAMPLES.

1. Divide $4\sqrt{3}$ by $3\sqrt[3]{4}$.

Here, $4\sqrt{3}=4\times3^{\frac{1}{2}}=4\times3^{\frac{3}{6}}=4\times(27)^{\frac{1}{6}}$

$3\sqrt[3]{4}=3\times4^{\frac{1}{3}}=3\times4^{\frac{2}{6}}=3\times(16)^{\frac{1}{6}}$

Whence, $4\sqrt{3}\div3\sqrt[3]{4}=\frac{4}{3}(\frac{27}{16})^{\frac{1}{6}}$.

2. Divide $\sqrt{a^2x-a^2x^2}$ by a.

Here, $a=\sqrt{a^2}$, $\therefore\ \sqrt{a^2x-a^2x}\div a=\sqrt{a^2x-a^2x^2}\div\sqrt{a^2}=\sqrt{x-x^2}$.

3. Divide $\sqrt{96a^2}$ by $4a$. *Ans.* $\sqrt{6}$.

4. Divide $\sqrt{1024a^2b}$ by $8a$. *Ans.* $4\sqrt{b}$.

5. Divide $\sqrt[3]{324a^2}$ by $\sqrt{9a}$. *Ans.* $\sqrt[6]{144a}$.

6. Divide $(\sqrt{72}+\sqrt{32}-4)$ by $\sqrt{8}$. *Ans.* $5-\sqrt{2}$.

7. Divide $\sqrt{ab^2-b^2c}$ by $\sqrt{a-c}$. *Ans.* b.

8. Divide $\sqrt[5]{64}$ by 2. *Ans.* $\sqrt[5]{2}$.

9. Divide $4\sqrt[3]{12}$ by $2\sqrt{3}$. *Ans.* $2\sqrt[6]{\frac{16}{3}}$.

10. Divide $\sqrt{12}$ by $\sqrt[3]{24}$. *Ans.* $\sqrt[6]{3}$.

11. Divide $(144a^3x^2)^{\frac{1}{10}}$ by $(6ax)^{\frac{1}{5}}$. *Ans.* $(4a)^{\frac{1}{10}}$.

12. Divide $(a+b)^{\frac{1}{2}}$ by $(a^2-b^2)^{\frac{1}{3}}$.

Ans. $\left(\dfrac{a+b}{(a-b)^2}\right)^{\frac{1}{6}}$.

EXTRACTION OF THE SQUARE ROOT OF A BINOMIAL SURD.

(**100.**) A binomial surd is a binomial, one or both of whose terms are surds. Thus, $a\pm\sqrt{b}$, $\sqrt{a}\pm\sqrt{b}$, $5\pm\sqrt{7}$, are binomial surds. The square root of a binomial surd may sometimes be found to be another binomial surd. For example, the square root of $6+\sqrt{20}$ is $1+2\sqrt{5}$. Before we can obtain a formula for extracting the square root of a binomial surd, it will be necessary to establish the following lemmas.

LEMMA I.

The square root of a quantity cannot consist of the sum or difference of two quantities, one of which is rational and the other irrational.

For, if it be possible, let us have the equation

$$\sqrt{a}=b\pm\sqrt{c} \qquad (1)$$

By squaring, $$a=b^2\pm 2b\sqrt{c}+c \qquad (2)$$

By transposing, &c., $\pm\sqrt{c}=\dfrac{a-b^2-c}{2b}$; that is, an irrational quantity equal to a rational quantity, which is absurd.

LEMMA II.

If we have any equation which has a rational and an irrational quantity in each member, the rational and the irrational quantities are equal each to each.

If we have the equation

$$a\pm\sqrt{b}=x\pm\sqrt{y}; \qquad (1)$$

Then will $$a=x \qquad (2)$$

And $$\pm\sqrt{b}=\pm\sqrt{y} \qquad (3)$$

If a be not equal to x, make $x=a+c$ (3)

Substituting this value of x in equation (1), we have

$$a\pm\sqrt{b}=a+c\pm\sqrt{y} \qquad (3)$$

$\therefore \quad \pm\sqrt{b}=c\pm\sqrt{y}$, which, by *Lemma* I., is impossible; whence, $a=x$, and $\sqrt{b}=\sqrt{y}$.

LEMMA III.

If $\sqrt{a+\sqrt{b}}=x+y$; *then* $\sqrt{a-\sqrt{b}}=x-y$; *where* x *and* y *are both supposed to be radicals of the second degree, or only one of them.*

For by squaring the equation,

$$\sqrt{a+\sqrt{b}}=x+y,$$

We have $$a+\sqrt{b}=x^2+2xy+y^2;$$

Whence $$\sqrt{b}=x^2+2xy+y^2-a$$

In this last equation, x^2, y^2 and a are rational quantities, and therefore $2xy$ must be an irrational quantity; for, if it were not, we should have an irrational quantity equal to a rational quantity. But, by Lemma I., there can be no such equality. Hence, $2xy$ is an irrational quantity, and by Lemma II., we have

$$a=x^2+y^2 \qquad (1)$$

and, $$\sqrt{b}=2xy \qquad (2)$$

Eq. (1)—Eq. (2) gives $$a-\sqrt{b}=x^2-2xy+y^2 \qquad (3)$$

Whence $$\sqrt{a-\sqrt{b}}=x-y \qquad (4)$$

(**101.**) We will now obtain a formula for extracting the square root of a binomial surd.

Assume $$x+y=\sqrt{a+\sqrt{b}} \qquad (1)$$

Then, by Lemma III., $$x-y=\sqrt{a-\sqrt{b}} \qquad (2)$$

Eq. (1) squared gives $$x^2+2xy+y^2=a+\sqrt{b} \qquad (3)$$

Eq. (2) squared gives $$x^2-2xy+y^2=a-\sqrt{b} \qquad (4)$$

By adding eq. (3) to eq. (4), and dividing by 2,

$$x^2+y^2=a \qquad (5)$$

Eq. (1)×eq. (2) gives $$x^2-y^2=\sqrt{x^2-b}=c^* \qquad (6)$$

Whence, from equations (5) and (6) $$x=\sqrt{\frac{a+c}{2}}, \qquad (7)$$

and $$y=\sqrt{\frac{a-c}{2}}. \qquad (8)$$

$$\therefore \quad x+y=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}, \qquad (9)$$

and $$x-y=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}. \qquad (10)$$

Whence $$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}, \qquad (A)$$

and $$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}} \qquad (B)$$

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* We make $\sqrt{a^2-b}=c$ for the sake of brevity.

If the extraction of the square root of a binomial surd is possible, c must be rational, and therefore a^2-b must be a perfect square. We will now apply formulas (A) and (B), in extracting the square roots of binomial surds.

EXAMPLES.

1. What is the square root of $11+\sqrt{72}$, or $11+6\sqrt{2}$?

Here $a=11$, $b=72$; and $c=\sqrt{a^2-b}=\sqrt{121-72}=7$. Therefore, $\sqrt{11+\sqrt{72}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}=3+\sqrt{2}$.

2. What is the square root of $7+4\sqrt{3}$?

Ans. $2+\sqrt{3}$.

3. What is the square root of $43-15\sqrt{8}$?

Ans. $5-3\sqrt{2}$.

4. What is the square root of $5-\sqrt{24}$?

Ans. $\sqrt{3}-\sqrt{2}$.

5. What is the square root of $3-2\sqrt{2}$?

Ans. $\sqrt{2}-1$.

6. What is the square root of $28+5\sqrt{12}$?

Ans. $5+\sqrt{3}$.

7. What is the square root of $2m+2\sqrt{m^2-n^2}$?

Ans. $\sqrt{m+n}+\sqrt{m-n}$.

8. What is the square root of $x-2\sqrt{x-1}$?

Ans. $\sqrt{x-1}-1$.

9. What is the square root of $28+10\sqrt{3}$?

Ans. $5+\sqrt{3}$.

10. What is the square root of $3-4\sqrt{-1}$?

Ans. $2-\sqrt{-1}$.

11. What is the square root of $18\pm2\sqrt{77}$?

Ans. $\sqrt{7}\pm\sqrt{11}$.

12. What is the square root of $94+42\sqrt{5}$?

Ans. $7+3\sqrt{5}$.

13. What is the square root of $m^2+n+2m\sqrt{n}$?

Ans. $m+\sqrt{n}$.

14. What is the square root of $42+3\sqrt{174\frac{2}{9}}$?

Ans. $\sqrt{28}+\sqrt{14}$.

IMAGINARY QUANTITIES.

(102.) Any expression which represents an even root of a negative quantity is called an *imaginary quantity.*

(103.) We have seen that an even root of a negative quantity is impossible, and therefore imaginary quantities have no real value; notwithstanding this circumstance, they are of much use in many parts of mathematical analysis, since, when they are employed according to the rules for operating upon them, they lead to important and possible results.

(104.) The addition and subtraction of imaginary quantities are performed by the ordinary rules for adding and subtracting radicals; but in the multiplication and division of them, we must pay attention to some important particulars which do not belong to other quantities.

(105.) In the first place, it is plain that $\sqrt{-a}\times\sqrt{-a}=-a$, since the square root of any quantity multiplied by itself must produce the quantity under the radical. But by the rule for the multiplication of quantities, $-a\times-a=a^2$. Therefore, $\sqrt{-a}\times\sqrt{-a}=\sqrt{a^2}$; whence, it follows that $\sqrt{a^2}=-a$. If we say that $\sqrt{a^2}=+a$, then we shall have $+a=-a$, as the conse-

quence. But this reasoning is incorrect, since the square root of a^2 cannot be $+a$ and $-a$ at the same time. In this case, we know that the square root of a^2 is $-a$, since a^2 was produced by multiplying $-a$ by $-a$.

(106.) If we have two imaginary quantities which are different, as $\sqrt{-a}$ and $\sqrt{-b}$, we do not at once perceive whether their product, $\sqrt{ab}$, is to be taken *negatively* or *positively*. But, by observing that $\sqrt{-a}=\sqrt{a}\times\sqrt{-1}$, and that $\sqrt{-b}=\sqrt{b}\times\sqrt{-1}$, their product is equal to $\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}=\sqrt{ab}\times(\sqrt{-1})^2=\sqrt{ab}\times-1=-\sqrt{ab}$. Hence, it appears that the proper sign of the product, $\sqrt{ab}$, is *minus*.

(107.) From what has been said, and from the principle in regard to the signs in multiplication, we may form the following table for the multiplication of imaginary quantities.

$$(+\sqrt{-a})\times(+\sqrt{-a})=-a \qquad (1)$$
$$(-\sqrt{-a})\times(-\sqrt{-a})=-a \qquad (2)$$
$$(+\sqrt{-a})\times(+\sqrt{-b})=-\sqrt{ab} \qquad (3)$$
$$(-\sqrt{-a})\times(-\sqrt{-b})=-\sqrt{ab} \qquad (4)$$
$$(+\sqrt{-a})\times(-\sqrt{-b})=+\sqrt{ab} \qquad (5)$$

EXAMPLES.

1. Multiply $3-\sqrt{-5}$ by $4-2\sqrt{-5}$.

Operation.

$$\begin{array}{l} 3-\ \sqrt{-5} \\ \underline{4-2\sqrt{-5}} \\ \left.\begin{array}{l} 12-4\sqrt{-5} \\ \quad\ -6\sqrt{-5}-10 \end{array}\right\}=2-10\sqrt{-5} \end{array}$$

2. Multiply $2\sqrt{3}-\sqrt{-5}$ by $4\sqrt{3}-2\sqrt{-5}$.

Ans. $14-8\sqrt{-15}$.

3. Multiply $9+6\sqrt{-1}$ by $3+7\sqrt{-1}$.

Ans. $-15+81\sqrt{-1}$.

4. Multiply $4\sqrt{-5}$ by $5\sqrt{-1}$. *Ans.* $-20\sqrt{5}$.

5. Multiply $-5\sqrt{-3}$ by $-3\sqrt{-4}$.

Ans. $-15\sqrt{12}$.

6. Multiply $8+\sqrt{-5}$ by $\sqrt{-3}$.

Ans. $8\sqrt{-3}-\sqrt{15}$.

7. Multiply $2+\sqrt{-3}$ by $4+\sqrt{-5}$.

Ans. $8+4\sqrt{-3}+2\sqrt{-5}-\sqrt{15}$.

(108.) The quotient of two radicals which have the same sign, and which arise from taking the square root of a negative quantity, is equal to the square root of their quotient. That is,

$$\frac{+\sqrt{-a}}{+\sqrt{-b}}=\frac{\sqrt{a}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=\sqrt{\frac{a}{b}}$$

And $$\frac{-\sqrt{-a}}{-\sqrt{-b}}=\frac{-\sqrt{a}\times\sqrt{-1}}{-\sqrt{b}\times\sqrt{-1}}=\sqrt{\frac{a}{b}}.$$

If the two imaginaries have not the same sign, it is plain that their quotient will be *minus* the square root of their quotient.

EXAMPLES.

1. Divide $10\sqrt{-14}$ by $2\sqrt{-7}$. *Ans.* $5\sqrt{2}$.

2. Divide $6+\sqrt{-2}$ by $6-\sqrt{-2}$.

Operation.

$$\frac{6+\sqrt{-2}}{6-\sqrt{-2}}$$

Multiply both the numerator and the denominator by the numerator, and we have

$$\frac{36+12\sqrt{-2}-2}{36+2}=\frac{17+6\sqrt{-2}}{19}$$

3. Divide $3+2\sqrt{-1}$ by $3-2\sqrt{-1}$.

Ans. $\frac{1}{13}(5+12\sqrt{-1})$.

4. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

Ans. $1+\sqrt{-2}$.

5. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* $\sqrt{-1}$.

6. Divide $-\sqrt{-1}$ by $-6\sqrt{-3}$.

Ans. $+\frac{1}{6\sqrt{3}}$.

CHAPTER VI.

ON THE SOLUTION OF PURE QUADRATICS AND OTHERS, WHICH MAY BE SOLVED WITHOUT COMPLETING THE SQUARE.

(109.) AN equation of the second degree, which contains only the second power of the unknown quantity, is called a pure quadratic. Thus, $ax^2-6x^2=c+10$ is a pure quadratic.

(110.) When the terms of an equation involve only the square of the unknown quantity, its value may be readily found, since we may so reduce the equation, that the square of the unknown quantity may constitute one member, and known quantities the other; then, by extracting the square root of each member, the value of the unknown quantity is determined. Since the square root of a quantity is positive or negative, it follows that the unknown quantity has two values in a pure quadratic. These two values are numerically equal, but they have contrary signs. In practical problems, it can generally be determined, which of these values or roots should be taken.

(111.) No general rule can be given for solving such examples as will be found in this chapter. The student must rely on his own ingenuity, and a *thorough* acquaintance with the preceding chapters. If he will carefully study the solutions which are given of the following examples, he will get an idea of the methods to be employed in reducing such equations.

EXAMPLES.

1. Given $\sqrt{x+16}=2+\sqrt{x}$ to find x.

By squaring each member, we have, $x+16=4+4\sqrt{x}+x$ (1)

By transposing in (1), $4\sqrt{x}=12$ (2)

Or, $\sqrt{x}=3$ (3)

Whence, $x=9$ (4)

2. Given $\dfrac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\dfrac{a}{b}$ to find x.

By clearing of fractions, we have,

$$b\sqrt{x}+b\sqrt{b}=a\sqrt{x}-a\sqrt{b} \qquad (1)$$

By transposing in (1), $(a-b).\sqrt{x}=\sqrt{b}.(a+b).$ (2)

By squaring (2), $(a-b)^2.x=b.(a+b)^2$; (3)

Whence, $x=\dfrac{b(a+b)^2}{(a-b)^2}$ (4)

3. Given $\dfrac{1}{x}+\dfrac{1}{a}=\sqrt{\dfrac{1}{a^2}+\sqrt{\dfrac{4}{a^2x^2}+\dfrac{9}{x^4}}}$ to find x.

By squaring each member, we have,

$$\frac{1}{x^2}+\frac{2}{ax}+\frac{1}{a^2}=\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}; \qquad (1)$$

Whence $\dfrac{1}{x^2}+\dfrac{2}{ax}=\sqrt{\dfrac{4}{a^2x^2}+\dfrac{9}{x^4}}$; (2)

Squaring(2), $\dfrac{1}{x^4}+\dfrac{4}{ax^3}+\dfrac{4}{a^2x^2}=\dfrac{4}{a^2x^2}+\dfrac{9}{x^4}.$ (3)

By transposing in (3), $\dfrac{4}{ax^3}=\dfrac{8}{x^4}$; (4)

Whence $x=2a$ (5)

4. Given $\left\{\begin{array}{ll} x^2-xy=54 & (1) \\ xy-y^2=18 & (2) \end{array}\right\}$ to find the values of x and y.

Eq. (1)−eq. (2) gives $x^2-2xy+y^2=36$ (3)

Sq. root of eq. (3) gives $x-y=6$ (4)

Eq. (1)+eq. (2) gives $x^2-y^2=72$ (5)

Eq. (5)÷eq. (4) gives $x+y=12$ (6)

Eq. (4)+eq. (6) gives $2x=18$ (7)

Eq. (6)−eq. (4) gives $2y=6$ (8)

Whence, $x=9$, and $y=3$

5. Given $\left\{\begin{array}{ll} \frac{1}{x}+\frac{1}{y}=\frac{1}{2} & (1) \\ \frac{2}{xy}=\frac{1}{9} & (2) \end{array}\right\}$ to find the values of x and y.

Eq. (1) squared gives	$\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{4}$	(3)
Eq. (2)×2 gives	$\frac{4}{xy}=\frac{2}{9}$	(4)
Eq. (3)−eq. (4) gives	$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{36}$	(5)
Sq. root of eq. (5) gives	$\frac{1}{x}-\frac{1}{y}=\frac{1}{6}$	(6)
Eq. (1)+eq. (6) gives	$\frac{2}{x}=\frac{2}{3}$	(7)
Eq. (1)−eq. (6) gives	$\frac{2}{y}=\frac{1}{3}$	(8)
Whence	$x=3$, and $y=6$	

6. Given $\left\{\begin{array}{ll} x^2+y^2+xy(x+y)=68 & (1) \\ x^3+y^3-3x^2-3y^2=12 & (2) \end{array}\right\}$ to find the values of x and y.

From eq. (1)	$x^2+y^2+x^2y+xy^2=68$	(3)
From eq. (2)	$x^3+y^3-3(x^2+y^2)=12$	(4)
Eq. (3)×3 gives	$3(x^2+y^2)+3x^2y+3xy^2=204$	(5)
Eq. (4)+(5)	$x^3+3x^2y+3xy^2+y^3=216$	(6)
Cube root of eq. (6) gives	$x+y=6$	(7)
Eq. (7) squared gives	$x^2+2xy+y^2=36$	(8)
The value of $x+y$ put in eq. (1) gives	$x^2+y^2+6xy=68$	(9)
Eq. (9)−eq. (8) gives	$4xy=32$	(10)
Eq. (8)−eq. (10) gives	$x^2-2xy+y^2=4$	(11)
Sq. root of eq. (11) gives	$x-y=2$	(12)
Eq. (7)+eq. (12) gives	$2x=8$	(13)
Ex. (7)−eq. (12) gives	$2y=4$	(14)

$\therefore \quad x=4$, and $y=2$

7. Given $\left\{\begin{array}{ll}\dfrac{x^2+xy+y^2}{x+y}=7 & (1)\\ \dfrac{x^2-xy+y^2}{x-y}=9 & (2)\end{array}\right\}$ to find the values of x and y.

Multiply both terms of the first fraction by $x-y$, and both terms of the second fraction $x+y$, and equations (1) and (2) become

$$\frac{x^3-y^3}{x^2-y^2}=7 \qquad (3)$$

and $$\frac{x^3+y^3}{x^2-y^2}=9 \qquad (4)$$

Eq. (3)+eq. (4) gives $$\frac{2x^3}{x^2-y^2}=16 \qquad (5)$$

Eq. (4)—eq. (3) gives $$\frac{2y^3}{x^2-y^2}=2 \qquad (6)$$

Eq. (5)÷(6) gives $$\frac{x^3}{y^3}=8 \qquad (7)$$

Whence, $$\frac{x}{y}=2 \qquad (8)$$

Whence, $x=2y$, $x^2=4y^2$, and $x^3=8y^3$. By substituting these values of x, x^2, and x^3, in (5), we have

$$\frac{16y^3}{4y^2-y^2}=16;$$

Whence, $y=3$, and $x=6$.

8. Given $\dfrac{\sqrt{a}-\sqrt{a-x}}{\sqrt{a}+\sqrt{a-x}}=a$, to find the value of x.

Multiply the numerator and denominator of the fraction by the numerator, and the equation becomes

$$\frac{(\sqrt{a}-\sqrt{a-x})^2}{x}=a, \qquad (1)$$

Or, $$(\sqrt{a}-\sqrt{a-x})^2=ax. \qquad (2)$$

Extracting sq. root, $$\sqrt{a}-\sqrt{a-x}=\sqrt{ax}. \qquad (3)$$

By transposing, $$\sqrt{a}-\sqrt{ax}=\sqrt{a-x}. \qquad (4)$$

By squaring (4), $a-2a\sqrt{x}+ax=a-x;$ (5)

Whence, $ax-2a\sqrt{x}=-x.$ (6)

Dividing by $x^{\frac{1}{2}}$, $a\sqrt{x}-2a=-\sqrt{x}.$ (7)

Or, $(a+1)\sqrt{x}=2a;$ (8)

Whence, $\sqrt{x}=\dfrac{2a}{a+1}.$ (9)

By squaring (9), $x=\dfrac{4a^2}{(a+1)^2}$ (10)

9. Given $\sqrt[5]{9x-4}+6=8$, to find the value of x.

Ans. $x=4.$

10. Given $\sqrt{x-32}=16-\sqrt{x}$, to find the value of x.

Ans. $x=81.$

11. Given $\sqrt{4x+21}=2\sqrt{x}+1$, to find the value of x.

Ans. $x=25.$

12. Given $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$, to find the value of x.

Ans. $x=6.$

13. Given $\dfrac{5x-9}{\sqrt{5x}+3}-1=\dfrac{\sqrt{5x}-3}{2}$, to find the value of x.

Ans. $x=5.$

14. Given $\dfrac{\sqrt{9x}-4}{\sqrt{x}+2}=\dfrac{15+\sqrt{9x}}{\sqrt{x}+40}$, to find the value of x.

Ans. $x=4.$

15. Given $\sqrt[m]{a+x}=\sqrt[2m]{x^2+5ax+b^2}$, to find the value of x.

Ans. $x=\dfrac{a^2-b^2}{3a}.$

16. Given $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$, to find the value of x.

Ans. $x=25.$

17. Given $\sqrt{2+x}+\sqrt{x}=\frac{4}{\sqrt{2+x}}$, to find the value of x.

Ans. $x=\frac{2}{3}$.

18. Given $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$, to find the value of x.

Ans. $x=4$.

19. Given $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\sqrt{\frac{x}{x+\sqrt{x}}}$, to find the value of x.

Ans. $x=\frac{25}{16}$.

20. Given $\frac{ax-b^2}{\sqrt{ax}+b}=c+\frac{\sqrt{ax}-b}{c}$, to find the value of x.

Ans. $x=\frac{1}{a}\left(b+\frac{c^2}{c-1}\right)^2$.

21. Given $\frac{\sqrt{x}+2a}{\sqrt{x}+b}=\frac{\sqrt{x}+4a}{\sqrt{x}+3b}$, to find the value of x.

Ans. $x=\left(\frac{ab}{a-b}\right)^2$.

22. Given $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$, to find the value of x.

Ans. $x=4$.

23. Given $\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}=\frac{x}{b}$, to find the value of x.

Ans. $x=\pm\sqrt{2ab-b^2}$.

24. Given $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$, to find the value of x.

Ans. $x=\pm\frac{a}{\sqrt{3}}$.

25. Given $\left\{\begin{array}{l} x^2+xy=12 \\ y^2+xy=24 \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 2$, $y=\pm 4$.

26. Given $\left\{\begin{array}{l} x^2-xy=48y \\ xy-y^2=\ \ 3x \end{array}\right\}$ to find the values of x and y.

Ans. $x=16$, or $-\frac{48}{5}$, $y=4$, or $\frac{12}{5}$.

27. Given $\left\{\begin{array}{l} x^{\frac{2}{3}}+y^{\frac{2}{3}}=13 \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=\ \ 5 \end{array}\right\}$ to find the values of x and y.

Ans. $x=27$, and $y=8$.

28. Given $\left\{\begin{array}{l} x^2y+xy^2=180 \\ x^3+\ y^3=189 \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, or 4; $y=4$, or 5.

29. Given $\left\{\begin{array}{l} x^3+y^3=(x+y)xy \\ x^2y+xy^2=4xy \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$, $y=2$.

30. Given $\left\{\begin{array}{l} x^2+y\sqrt{xy}=\ \ 9 \\ y^2+x\sqrt{xy}=18 \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 1$, $y=\pm 4$.

31. Given $\left\{\begin{array}{l} x^4y^3-x^3y^4=216 \\ x^2y\ -xy^2\ =\ \ \ \ 6 \end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, or -2, $y=2$, or -3.

32. Given $\left\{\begin{array}{l} xy(x+y)=\ \ \ 84 \\ x^2y^2(x^2+y^2)=3600 \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, or 3, $y=3$, or 4.

33. Given $\left\{\begin{array}{l} x^2+x\sqrt[3]{xy^2}=\ \ 208 \\ y^2+y\sqrt[3]{x^2y}=1053 \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 8$, $y=\pm 27$.

34. Given $\left\{\begin{array}{l} x^3-y^3=56 \\ x-y=\dfrac{16}{xy} \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, or -2, $y=2$, or -4.

35. Given $\left\{\begin{array}{l} x^{\frac{1}{3}}+y^{\frac{1}{3}}=6 \\ x+y=72 \end{array}\right\}$ to find the values of x and y.*

Ans. $x=64$, or 8, $y=8$, or 64.

36. Given $\left\{\left(\frac{2x+3}{2x-3}\right)^{\frac{1}{3}}+\left(\frac{2x-3}{2x+3}\right)^{\frac{1}{3}}=\frac{8}{13}\cdot\left(\frac{4x^2+9}{4x^2-9}\right)\right\}$ to find the value of x. *Ans.* $x=\frac{27}{14}$.

37. Given $\left\{\frac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9\right\}$ to find the value of x.

Ans. $x=\frac{4}{9}$.

38. Given $x^2+3x-7=x+2+\frac{18}{x}$, to find the values of x.

Ans. $x=\pm 3$.

39. Given $\frac{\sqrt{x+a}}{\sqrt{x}}+\frac{2\sqrt{a}}{\sqrt{x+a}}=b^2.\frac{\sqrt{x}}{\sqrt{x+a}}$, to find the values of x.

Ans. $x=\frac{a}{(b\pm 1)^2}$.

40. Given $\frac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\frac{n^2a}{x-a}$, to find the values of x.

Ans. $x=\frac{a(1\pm n)^2}{1\pm 2n}$.

41. Given $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$, to find the values of x.

Ans. $x=\frac{2ab}{b^2+1}$.

42. Given $\frac{a+x+\sqrt{x^2+2ax}}{a+x}=b$, to find the values of a.

Ans. $x=\pm a.\frac{1\pm\sqrt{2b-b^2}}{\sqrt{2b-b^2}}$.

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* **Note.—In this and in some of the preceding examples, we may avoid fractional exponents, by substitution. Thus, let $m=x^{\frac{1}{3}}$, and $n=y^{\frac{1}{3}}$; whence, $m^3=x$, $n^3=y$.**

43. Given $\left\{\begin{array}{c} 3xy-3x=y^3-y^2 \\ y^2+x=4 \end{array}\right\}$ to find the values of x and y.

Ans. $x=1$, $y=\pm\sqrt{3}$.

44. Given $\left\{\begin{array}{l} (x^2-y^2)\times(x-y) \ =3xy \\ (x^4-y^4)\times(x^2-y^2)=45x^2y^2 \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, or 2, $y=2$, or 4.

45. Given $\left\{\begin{array}{c} x+\sqrt{xy}+y=19 \\ x^2+xy+y^2=133 \end{array}\right\}$ to find the values of x and y.

Ans. $x=9$, or 4, $y=4$, or 9.

46. Given $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\dfrac{x}{a}$, to find the real value of x.

Ans. $x=a$.*

47. Given $\left\{\begin{array}{l} 2(x^2+y^2)\times(x+y) \ =15xy \\ 4(x^4-y^4)\times(x^2+y^2)=75x^2y^2 \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$, $y=1$.

48. Given $\left\{\begin{array}{l} (x^2+y^2)\times(x+y)=2336 \\ (x^2-y^2)\times(x-y)=576 \end{array}\right\}$ to find the values of x and y.

Ans. $x=11$, or 5, $y=5$, or 11.

49. Given $\left\{\begin{array}{l} x^2+2y^2=x\sqrt{y}+2xy^{\frac{3}{2}} \\ x^2-2y^2=256-x\sqrt{y} \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 16$, $y=4$.

50. Given $\left\{\begin{array}{l} \dfrac{9}{8}\cdot\dfrac{(x+y)^{\frac{1}{3}}}{y}+\dfrac{9}{8}\cdot\dfrac{(x+y)^{\frac{1}{3}}}{x}=\dfrac{8}{7} \\ \dfrac{7}{4}\cdot\dfrac{(x-y)^{\frac{1}{3}}}{y}-\dfrac{7}{4}\cdot\dfrac{(x-y)^{\frac{1}{3}}}{x}=\dfrac{1}{9} \end{array}\right\}$ to find the values of x and y.

Ans. $x=\frac{9}{2}$, $y=\frac{7}{2}$.

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* There are two other roots of x, both of which are imaginary. In Hutton's Mathematics the imaginary roots only are given.

PROBLEMS IN PURE QUADRATICS.

(112.) In solving problems in pure quadratics, it will be necessary to recollect that, in many cases, the product of two or more factors must only be expressed. Much also depends on the notation which is adopted. To show the necessity of these observations, we will give the solutions of a few problems.

1. Two workmen, A and B, were engaged to work for a certain number of days at different rates. At the end of the time, A, who had played 4 of those days, had 75 shillings to receive; but B, who had played 7 of those days, received only 48 shillings. Now, had B only played 4 days, and A played 7 days, each would have received the same sum. For how many days were they engaged; and how many did each work, and what had each per day?

Let x=the number of days for which they were engaged;

$\therefore$ $x-4$=the number A worked,

And $x-7$=the number B worked,

And $\dfrac{75}{x-4}$=the number of shillings A received per day.

And $\dfrac{48}{x-7}$=the number of shillings B received per day.

$\therefore$ $\dfrac{75(x-7)}{x-4}$=the number of shillings A received in $(x-7)$ days.

And $\dfrac{48(x-4)}{x-7}$=the number of shillings B rec'd in $(x-4)$ days;

Whence, $$\frac{75(x-7)}{x-4}=\frac{48(x-4)}{x-7} \quad (1)$$

Or, $$75(x-7)^2=48(x-4)^2 \quad (2)$$

Dividing by 3, $$25(x-7)^2=16(x-4)^2 \quad (3)$$

Extracting sq. root, $$5(x-7)=4(x-4) \quad (4)$$

$\therefore$ $x=19$; hence they were engaged to work 19 days, A worked 15, and B 7 days; B received 4 shillings per day, and A 5 shillings per day. If we had actually performed the multiplications indicated in equation (1), then cleared it of fractions, transposed and united its terms, we

should have obtained an equation which could not be reduced by any method yet considered.

2. It is required to find two numbers such, that the product of the greater and square root of the less may be equal to 48, and the product of the less and square root of the greater may be 36.

Let x^2 and y^2 be the two numbers.

$$\therefore \quad x^2y=48 \qquad (1)$$

$$\text{and } xy^2=36 \qquad (2)$$

$$\text{Eq. (1)} \div \text{eq. (2) gives } \quad \frac{x}{y}=\tfrac{4}{3} \qquad (3)$$

$$y=\tfrac{3}{4}x \qquad (4) \text{ from (3)}$$

Substitute this value of y in (1), and it becomes,

$$\tfrac{3}{4}x^3=48 \qquad (5)$$

$$\therefore \quad x=4 \qquad (6)$$

$$y=3 \qquad (7)$$

Hence the numbers are 16 and 9.

3. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw at each time?

Let $a=256$,

and x=number of gallons drawn the first time.

$\therefore$ $a-x$= " " left in the cask.

The cask is now filled with water, and in a gallons of the mixture thus formed, there are $(a-x)$ gallons of pure wine. In one gallon of the mixture there is one ath of this quantity, and in x gallons of the mixture, the number drawn at each draught, there are x times one ath of $\overline{a-x}$ gallons.

Hence, $(a-x)\dfrac{x}{a}=\dfrac{ax-x^2}{a}$=the number of gallons drawn the 2d time;

And $x+\dfrac{ax-x^2}{a}=\dfrac{2ax-x^2}{a}$=the number drawn the 1st and 2d time.

Whence $a-\frac{2ax-x^2}{a}=\frac{a^2-2ax+x^2}{a}$ = the number of gallons left after the 2d draught.

Reasoning as before, we find that,

$\frac{a^2-2ax+x^2}{a}\times\frac{x}{a}=\frac{a^2x-2ax^2+x^3}{a^2}$ = the number of gallons drawn the 3d time. Whence,

$$a-\left(x+\frac{ax-x^2}{a}+\frac{a^2x-2ax^2+x^3}{a^2}\right)=\frac{a^3-3a^2x+3ax^2-x^3}{a^2}$$

= the number of gallons left in the cask after the third draught.

Therefore, $\frac{a^3-3a^2x+3ax^2-x^3}{a^2}\times\frac{x}{a}=\frac{a^3x-3a^2x^2+3ax^3-x^4}{a^3}$ = the number of gallons drawn the 4th time. Whence,

$$a-\left(x+\frac{ax-x^2}{a}+\frac{a^2x-2ax^2+x^3}{a^2}+\frac{a^3x-3a^2x^2+3ax^3-x^4}{a^3}\right)=$$

$\frac{a^4-4a^3x+6a^2x^2-4ax^3+x^4}{a^3}$ = the number of gallons left after the 4th draught. Therefore, by the question we have,

$$\frac{a^4-4a^3x+6a^2x^2-4ax^3+x^4}{a^3}=81 \qquad (1)$$

$$\therefore a^4-4a^3x+6a^2x^2-4ax^3+x^4=81a^3 \qquad (2)$$

Extracting 4th root of eq. (2), $a-x=3a^{\frac{3}{4}}$ (3)

Or, $256-x=3.256^{\frac{3}{4}}=3.64=192$ (4)

Whence, $x=64$,

and the quantities drawn off at each time were 64, 48, 36, and 27 gallons.

4. What two numbers are those, whose sum is to the greater as 10 to 7; and whose sum multiplied by the less produces 270?

Ans. ± 21, and ± 9.

5. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Ans. ± 18, and ± 14.

6. There are two numbers in the proportion of 4 to 5, the difference of whose squares is 81. What are the numbers?

Ans. ± 12, and 15.

7. A mercer bought a piece of silk for 324 shillings; and the number of shillings which he paid for a yard was to the number of yards as 4 to 9. How many yards did he buy, and what was the price per yard? *Ans.* 27 yds. at 12*s.* per yd.

8. It is required to divide the number 18 into two such parts, that 25 times the square of less part may be equal to 16 times the square of the greater part. What are the parts?

Ans. 10 and 8.

9. A number of boys set out to rob an orchard, each carrying as many bags as there were boys in all, and each bag capable of containing 4 times as many apples as there were boys. They filled their bags and found that the number of apples was 2916. How many boys were there? *Ans.* 9 boys.

10. From two towns, C and D, which were at the distance of 396 miles, two persons, A and B, set out at the same time, and met each other, after travelling as many days as are equal to the difference of the number of miles they travelled per day; when it appears that A has travelled 216 miles. How many miles did each travel per day? *Ans.* A went 36, B 30.

11. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number? *Ans.* 23.

12. There are two rectangular vats, the greater of which contains 20 solid feet more than the other. Their capacities are in the ratio of 4 to 5; and their bases are squares, the side of each of which is equal to the depth of the other. What are the depths?

Ans. 5 feet, and 4 feet.

13. Bought two square carpets for £62 1*s.*; for each of which I paid as many shillings per yard as there were yards in its side. Now had each of them cost as many shillings per yard as there were yards in the side of the other, I should have paid 17*s.* less. What was the size of each?

Ans. One contained 81, the other 64 square yards.

14. A and B are two towns, situated on the bank of a river which runs at the rate of 4 miles per hour. A waterman rows from A to B and back again, and finds that he is 39 minutes longer upon the water than he would have been had there been no stream. The next day he repeats his voyage with another waterman, with whose assistance he can row half as fast again; and they find that they are only 8 minutes longer in performing their voyage than they would have been had there been no stream. Determine the rate at which the waterman would row by himself. *Ans.* 6 miles per hour.

15. A person bought a number of apples and pears amounting together to 80. Now the apples cost twice as much as the pears: but had he bought as many apples as he did pears, and as many pears as he did apples, his apples would have cost 10 cents, and his pears 45 cents. How many did he buy of each?

Ans. 60 apples and 20 pears.

16. A gentleman exchanged a quantity of brandy for a quantity of rum and £11 5*s.*; the brandy and rum being each valued at as many shillings per gallon as there were gallons of that liquor. Now had the rum been worth as many shillings per gallon as the brandy was, the whole value of the rum and brandy would have been £56 5*s.* How many gallons were there of each? *Ans.* 25 gallons of brandy, and 20 of rum.

17. A and B carried 100 eggs to market, and each received the same sum. If A had carried as many as B, he would have received 18 pence for them, and if B had taken as many as A, he would have received only 8 pence for them. How many had each? *Ans.* A had 40, B 60.

18. A farmer has two cubical stacks of hay. The side of one is 3 yards longer than the side of the other; and the difference of their contents is 117 solid yards. Required the side of each.

Ans. The side of one is 5 yds., that of the other 2.

19. What two numbers are those, whose difference multiplied by the greater produces 40, and by the less 15?

Ans. ± 8 and ± 3.

20. What two numbers are those, whose difference multiplied by the less produces 42, and by their sum 133?

Ans. ± 13 and ± 6.

21. Find two numbers which are to each other as 5 to 8, and whose product is 360. *Ans.* ± 15 and ± 24.

22. There is a field in the form of a rectangular parallelogram, whose length is to its breadth as 6 to 5. A part of this, equal to one sixth of the whole, being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

Ans. The sides are 30, and 25 yds.

23. The paving of two square yards cost £205; a yard of each costing one fourth of as many shillings as there were yards in the side of the other. And a side of the greater and less together measures 41 yards. Required the length of a side of each?

Ans. 25, and 16 yards.

24. Two traders, A and B, set out to meet each other, A leaving the town C at the same time that B left D. They travelled the direct road C D, and on meeting it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What was the distance between C and D?

Ans. 126 miles.

25. A and B lay out some money on speculation. A disposes of his bargain for \$11, and gains as much per cent. as B lays out; B's gain is \$36, and it appears that A gains four times as much per cent. as B. Required the capital of each.

Ans. B's capital \$120, A's \$5.

26. The captain of a privateer descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her, and then observing the trader steering in a direction perpendicular to her former course, changed his own course so as to overtake her without making another tack. On comparing their reckonings, it was found that the privateer had run at the rate of 10 knots in an hour, and the trading vessel at the rate of 8 knots in the same time. Required the distance sailed by the privateer.

Ans. 25 miles.

27. A merchant laid out a certain sum on speculation, and found at the end of a year that he had gained a dollars. This he added to his stock, and at the end of the second year found that he had gained exactly as much per cent. as in the year preceding. Proceeding in the same manner, and each year adding to his stock the gain of the preceding year, he found that at the end of the fourth year his stock was equal to eighty-one sixteenths of his original stock. What was his original stock?

Ans. $2a$ dollars.

CHAPTER VII.

ADFECTED QUADRATICS WHICH INVOLVE ONE UNKNOWN QUANTITY.

(113.) An Adfected Quadratic is one that contains only the square and the first power of the unknown quantity. Thus, $ax^2+bx=c$ is an adfected quadratic equation. Every adfected quadratic equation may be reduced to the form of $ax^2+bx=c$.

(114.) Since $(x\pm a)^2=x^2\pm 2ax+a^2$, we may infer the following propositions, by the aid of which we may obtain one of the three terms of the square of a binomial by means of the other two.

I. *The first term of the square of a binomial may be obtained by dividing the second term by twice the square root of the third, and then squaring the quotient.*

II. *The second term may be obtained by taking twice the product of the square roots of the first and second terms, and prefixing to it the double sign.*

III. *The third term may be obtained by dividing the second term by twice the square root of the first term, and then squaring the quotient.*

(115.) If, in the expression $x^2\pm 2ax$, we regard $2a$ as the co-efficient of x, we see that $x^2\pm 2ax$ may be rendered a complete square by *adding to it the square of one half the co-efficient of* x.

If we have the quadratic equation $x^2+8x=20$, we can render the first member a complete square by adding to it 16; but if we add 16 to one member, we must add the same quantity to the other member, in order to preserve the equality. Hence, we may have, $x^2+8x+16=36$. By extracting the square root of this

equation, we have $x+4=6$, or $x=2$. Hence, it appears that in order to solve an adfected quadratic equation, the first member must be rendered a complete square, and then, by extracting the square root of each member of the resulting equation, a simple equation is obtained from which the value of the unknown quantity is readily found.

As the preceding propositions will be of use in reducing some equations, we will add a few examples for the purpose of making the student familiar with them. We shall write a cipher in the place of that term which is to be found, in order that we may have the square of a binomial.

1. Render $81x^2+0+\frac{1}{x^2}$ a complete square.

Here the second term is wanting. By Prop. II. the second term $=9x\times\frac{1}{x}\times2=18$. Therefore, $81x^2+18+\frac{1}{x^2}$ is a complete square.

2. Render $x^2+12x+0$ a complete square.

Ans. $x^2+12x+36$.

3. Render $4x^2+4x+0$ a complete square.

Ans. $4x^2+4x+1$.

4. Render $x^4+\frac{17}{2}x^3+0$ a complete square.

Ans. $x^4+\frac{17}{2}x^3+\frac{289x^2}{16}$.

5. Render $49x^2+98+0$ a complete square.

Ans. $49x^2+98+\frac{49}{x^2}$.

6. Render y^4+12xy^2+0 a complete square.

Ans. $y^4+12xy^2+36x^2$.

7. Render $25x^2+30x+0$ a complete square.

Ans. $25x^2+30x+9$.

8. Render $81+39x+0$ a complete square.

Ans. $81+39x+\frac{169x^2}{36}$.

9. Render $x^2y^2+2xy+0$ a complete square.

Ans. $x^2y^2+2xy+1$.

10. Render $4x^2+72x+0$ a complete square.

Ans. $4x^2+72x+324$.

(**116.**) If we represent the sum of all the known quantities in an adfected quadratic equation by c, the sum of all the co-efficients of the first power of the unknown quantity by b, and the sum of all the co-efficients of the second power by a, the equation will then take this form,

$$ax^2+bx=c \qquad \text{(A)}$$

In order to avoid fractions, we will multiply equation (A) by $4a$, and it becomes

$$4a^2x^2+4abx=4ac \qquad \text{(B)}$$

For the purpose of rendering the first member a perfect square, we find by Prop. I., in the preceding article, that we must add b^2 to each member. Equation (B) then becomes

$$4a^2x^2+4abx+b^2=4ac+b^2. \qquad \text{(C)*}$$

By extracting the square root of equation (C), we have,

$$2ax+b=\pm\sqrt{4ac+b^2}; \qquad \text{(D)}$$

Whence,
$$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}. \qquad \text{(E)}$$

From formula (E) we may derive the following rule for writing out the value of the unknown quantity.

RULE I.

Reduce the equation to the form of equation (A). Then the value of the unknown quantity is equal to the co-efficient of the first power of the unknown quantity taken with a contrary sign, plus or minus the square root of four times the product of the co-efficient of the highest power of the unknown quantity and the

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* NOTE.—The process of rendering the first member a perfect square is called "completing the square."

term independent of the unknown quantity, increased by the square of the co-efficient of the first power of the unknown quantity, and the whole divided by twice the co-efficient of the second power of the unknown quantity.

(**117.**) If we divide equation (A) in the preceding article by a, and then substitute for $\frac{b}{a}$, $2p$, and for $\frac{c}{a}$, q, it becomes

$$x^2+2px=q \qquad \text{(F)}$$

Add p^2 to each member of this equation, and it becomes

$$x^2+2px+p^2=p^2+q$$

By extracting the square root, we have,

$$x+p=\pm\sqrt{p^2+q};$$

Whence, $$x=-p\pm\sqrt{p^2+q} \qquad \text{(G)}$$

Formula (G) furnishes, by translating it into common language, the following rule.

RULE II.

Reduce the equation to the form of equation (F). *Then the value of the unknown quantity is equal to one half of the co-efficient of its first power, taken with a contrary sign, plus or minus the square root of the square of one half the co-efficient of the first power of the unknown quantity, increased by the term which is independent of the unknown quantity.*

(**118.**) If we substitute $2p$ for b, in equation (A), Art. 116, and then multiply each member by a, and add p^2 to each member of this last equation, it becomes

$$a^2x^2+2apx+p^2=ac+p^2$$

By extracting the square root of this equation, we have,

$$ax+p=\pm\sqrt{ac+p^2};$$

Whence, $$x=\frac{-p\pm\sqrt{ac+p^2}}{a} \qquad \text{(H)}$$

By translating formula (H) into common language, we have the following rule.

RULE III.

The value of the unknown quantity is equal to one half of the co-efficient of the first power of the unknown quantity taken with a contrary sign, plus or minus the square root of the product of the co-efficient of the highest power of the unknown quantity and the term independent of the unknown quantity, increased by the square of one half the co-efficient of the first power of the unknown quantity, and the whole divided by the co-efficient of the second power of the unknown quantity.

(**119.**) Before applying Rule I. or Rule III., we should divide each member of the equation by the greatest common divisor of the co-efficient of the first and second powers of the unknown quantity, and the term which is independent of the unknown quantity. Thus, in the equation $9x^2+15x=66$, we should, before applying the rule, divide each member by 3, the greatest common measure of 9, 15, and 66. If, after an equation is reduced in this manner, the co-efficient of the first power of the unknown quantity is an *even* number, Rule III. should be employed, if it is not an *even* number, Rule I. should be employed. Rule II. should be used when the co-efficient of the second power is unity, and that of the first power is an *even* number.

(**120.**) By examining formulas (E), (G), and (H), we find, that by using the *plus* sign before the radical, we shall obtain one value of the unknown quantity, and that by using the *minus* sign we shall obtain another value. Hence,

In every quadratic equation the unknown quantity may have two values.

Either of these values will, when substituted for the unknown quantity, satisfy the equation from which it was obtained.

In a quadratic equation the unknown quantity cannot have more than two values. For, if possible, suppose that the equation $ax^2+px=q$ has three distinct values, and represent these values by r, s, and t. Then each of these values must satisfy the equation. Hence, we have

$$ar^2+pr=q, \qquad (1)$$
$$as^2+ps=q, \qquad (2)$$
and $$at^2+pt=q. \qquad (3)$$

(Eq. (1)—eq. (2) gives $a(r^2-s^2)+p(r-s)=0$ (4)
Eq. (1)—eq. (3) gives $a(r^2-t^2)+p(r-t)=0$ (5)
Eq. (4) divided by $r-s$, gives $a(r+s)+p=0$ (6)
Eq. (5) divided by $r-t$, gives $a(r+t)+p=0$ (7)
Eq. (6)—eq. (7) gives $a(s-t)=0$ (8)

Now the last equation can only be satisfied by making $a=0$, or $s-t=0$. But a cannot equal 0, for then the proposed equation would not be a quadratic; hence, $s-t=0$, or $s=t$. Therefore, in a quadratic equation the unknown quantity cannot have three distinct values.

(121.) The rules which have been given for solving quadratics, will also enable us to solve any equation which can be reduced to the form $x^{2n}+2ax^n=c$; that is, any equation which contains the unknown quantity in two terms, and having the index of the unknown quantity in one term, double its index in the other. It must be observed, however, that we can seldom obtain all the roots in this manner.*

EXAMPLES.

1. Given $6x+\frac{35-3x}{x}=44$, to find the values of x.

Clearing of fractions, $6x^2+35-3x=44x$

By transposing, $6x^2-47x=-35$

Therefore, (Rule I.) $x=\frac{47\pm\sqrt{47^2-4\times6\times35}}{12}=\frac{47\pm37}{12}=7$, or $\frac{5}{6}$.

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* NOTE.—Every equation which contains only one unknown quantity has as many roots as there are units in the highest power of the unknown quantity. Such equations will be treated of in another part of this work.

2. Given $9x-4x^2+\sqrt{4x^2-9x+11}=5$, to find the values of x.

Add -11 to each member of the equation, and then change its signs, and it becomes

$$4x^2-9x+11-\sqrt{4x^2-9x+11}=6 \qquad (1)$$

Let $y=\sqrt{4x^2-9x+11}$; whence, $y^2=4x^2-9x+11$.

By substituting y and y^2 in place of their values in equation (1), we have

$$y^2-y=6 \qquad (2)$$

Therefore, (Rule I.) $y=\dfrac{1\pm\sqrt{25}}{2}=3$ or -2 $\qquad (3)$

$$\therefore \quad y^2=9, \text{ or } 4 \qquad (4)$$

$$\therefore \quad 4x^2-9x+11=9 \qquad (5)$$

$$\text{And} \quad 4x^2-9x+11=4 \qquad (6)$$

From eq. (5), we find that $x=2$, or $\frac{1}{4}$;

From eq. (6), we find that $x=\dfrac{9\pm\sqrt{-31}}{8}$

3. Given $x^4+\dfrac{13x^3}{2}-39x=81$, to find the values of x.

By transposition, $x^4+\dfrac{13x^3}{3}=81+39x$ $\qquad (1)$

By Prop. I., Art. 114, we find that we can render each member of equation (1) a perfect square, by adding $\dfrac{169x^2}{36}$ to each member. Hence, we have

$$x^4+\frac{13x^3}{3}+\frac{169x^2}{36}=81+39x+\frac{169x^2}{36} \qquad (2)$$

By extracting the square root of equation (2),

$$x^2+\frac{13x}{6}=9+\frac{13x}{6} \qquad (3)$$

$$\therefore \quad x=\pm 3$$

4. Given $\sqrt{x}-\dfrac{8}{x}=\dfrac{7}{\sqrt{x}-2}$, to find the values of x.

Let $y=\sqrt{x}$, then $y^2=x$. Substitute y^2 and y in the place of their values in the given equation, and it becomes

$$y-\frac{8}{y^2}=\frac{7}{y-2} \qquad (1)$$

By clearing this equation of fractions and transposing, it becomes

$$y^4-2y^3-7y^2-8y+16=0 \qquad (2)$$

If we proceed to extract the square root of equation (2), we shall discover that we can render the first member a perfect square by adding to it $16y^2$. Therefore, add $16y^2$ to each member of equation (2), and we have

$$y^4-2y^3+9y^2-8y+16=16y^2 \qquad (3)$$

By extracting square root, $\qquad y^2-y+\ 4=\pm 4y \qquad (4)$

$$\therefore \quad y^2-5y=-4\,; \qquad (5)$$

Whence, $\qquad y=4 \text{ or } 1 \qquad (6)$

$$\therefore \quad y^2=16 \text{ or } 1 \qquad (7)$$

And $\quad x=y^2=16 \text{ or } 1 \qquad (8)$

By using $-4y$ instead of $+4y$, we shall find that the other roots of x are $\frac{1+3\sqrt{-7}}{2}$, and $\frac{1-3\sqrt{-7}}{2}$, both being imaginary.

5. Given $\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\frac{x}{a}$, to find the values of x.

Multiply the numerator and denominator of the first member by the numerator, and the equation becomes

$$\frac{\left(x+\sqrt{x^2-a^2}\right)^2}{a^2}=\frac{x}{a} \qquad (1)$$

Multiply each member by a^2, and then extract the square root, and we have

$$x+\sqrt{x^2-a^2}=\pm\sqrt{ax} \qquad (2)$$

By transposing, $\qquad \sqrt{x^2-a^2}=\pm\sqrt{ax}-x \qquad (3)$

By squaring each member, $\quad x^2-a^2=ax\pm 2x\sqrt{ax}+x^2 \quad (4)$

By transposing and using the negative sign of $2x\sqrt{ax}$, we have

$$-ax+2x\sqrt{ax}+x^2=x^2+a^2 \qquad (5)$$

Add $2ax$ to each member of eq. (5),

$$ax+2x\sqrt{ax}+x^2=x^2+2ax+a^2 \quad (6)$$

Extracting sq. root of (6), $\sqrt{ax}+x=x+a \quad (7)$

$$\therefore \quad \sqrt{ax}=a; \quad (8)$$

Or, $ax=a^2$, (9)

And, $x=a$ (10)

If we take the negative sign of $2x\sqrt{ax}$, equation (6) may be written,

$$ax-2x\sqrt{ax}+x^2=x^2+2ax+a^2 \quad (11)$$

Extracting sq. root of (11), $\sqrt{ax}-x=x+a$ (12)

Or, $\sqrt{ax}=2x+a$ (13)

By squaring (13), $ax=4x^2+4ax+a^2$ (14)

$$\therefore \quad 4x^2+3ax=-a^2 \quad (15)$$

Whence, (Rule I.), $x=\dfrac{a(\pm\sqrt{-7}-3)^*}{8}$ (16)

6. Given $x^3-8x^2+19x-12=0$, to find the values of x.

Multiply each member by x, in order to make the first term, x^3, a perfect square, and the equation becomes

$$x^4-8x^3+19x^2-12x=0 \quad (1)$$

If we endeavor to extract the square root of the first member, we shall find that the first two terms of the root are x^2 and $-4x$; and that the remainder, $3x^2-12x$, may be written $3(x^2-4x)$. Hence, equation (1) may take this form,

$$(x^2-4x)^2+3(x^2-4x)=0 \quad (2)$$

Let $y=x^2-4x$, and then $y^2=(x^2-4x)^2$. Substitute y and y^2 in the place of their values in equation (2), and we have

$$y^2+3y=0, \quad (3)$$

Or, $y^2=-3y$; (4)

Whence $y=-3$.

$$\therefore \quad x^2-4x=-3;$$

Whence, $x=2\pm1=3$, or **1**.

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* Note.—In Hutton's Mathematics, from which this example is taken, only the imaginary roots are given.

If we divide each member of the given equation by $x-1$, we shall obtain an equation from which the other root, 4, may be found.

7. Given $\frac{y^3-10y^2+1}{y^2-6y+9}=y-3$, to find the values of y.

By clearing of fractions, we have,

$$y^3-10y^2+1=y^3-9y^2+27y-27. \qquad (1)$$

By transposition, $y^2+27y=28=27+1 \qquad (2)$

By Rule I., $y=\frac{-27\pm\sqrt{27^2+4\times27+4}}{2}, \qquad (3)$*

Or, $y=\frac{-27\pm(27+2)}{2}=1$, or -28.

8. Given $45x^2+64x=308$, to find the values of x.

Let $a=45$, and $c=26$. Then $2a-c=64$, and $8a-2c=308$. Substitute a, $2a-c$, and $8a-2c$ in the place of their values, and the equation becomes

$$ax^2+(2a-c)x=8a-c. \qquad (2)$$

By Rule I., $x=\frac{-2a+c\pm\sqrt{36a^2-12ac+c^2}}{2a}, \qquad (3)$

Or, $x=\frac{-2a+c\pm(6a-c)}{2a}; \qquad (4)$

Whence, $x=3$, or $-\frac{109}{45} \qquad (5)$

We may generally avoid a great deal of numerical calculation by *judiciously* substituting letters for numerals in those examples in which the value of the unknown quantity is an integer. The student should exercise his ingenuity in this way.

9. Given $133x^2-218x=543$, to find the values of x.

Let $a=133$, and $c=48$. Then, $2a-c=218$, and $3a+3c=543$. Substitute a, $2a-c$, and $3a+3c$ in the place of their values in the given equation, and it becomes

$$ax^2-(2a-c)x=3a+3c \qquad (1)$$

---

* Let the student observe that the quantity under the radical sign is the square of a binomial.

By Rule I., $$x=\frac{+2a-c\pm\sqrt{16a^2+8ac+c^2}}{2a}, \qquad (2)$$

Or, $$x=\frac{+2a-c\pm(4a+c)}{2a}; \qquad (3)$$

Whence, $$x=3, \text{ or } -\frac{181}{133} \qquad (4)$$

10. Given $3x^2+37x=876$, to find the values of x.

Let $a=37$. Then $23a+25=876$. Substitute a, and $23a+25$ in the place of their values, and the equation becomes

$$3x^2+ax=23a+25. \qquad (1)$$

Multiply (1) by 3, $$9x^2+3ax=69a+75. \qquad (2)$$

Since $a=37$, $75=2a+1$. Therefore $69a+75=69a+2a+1=71a+1$. Observe that $74a=2\times37\times a=2a^2$. Now, if from $71a+1$, we subtract $74a$, and add to the remainder its equal, $2a^2$, we shall have, $71a+1=2a^2-3a+1$. Therefore, $69a+75=2a^2-3a+1$. Hence, equation (2) may be written

$$9x^2+3ax=2a^2-3a+1. \qquad (3)$$

We find by Prop. I., Art. 114, that we must add $\frac{a^2}{4}$ to each member of equation (3), in order to render each a perfect square. Therefore, we have,

$$9x^2+3ax+\frac{a^2}{4}=\frac{9a^2}{4}-3a+1 \qquad (4)$$

Extracting square root, $$3x+\frac{a}{2}=\pm\left(\frac{3a}{2}-1\right), \qquad (5)$$

$\therefore$ $3x=a-1=36$, or $-2a+1=-73$;

Whence, $x=12$, or $-24\frac{1}{3}$.

11. Given $x^2+4x=140$, to find the values of x.

Ans. $x=10$, or -14.

12. Given $x^2-6x+8=80$, to find the values of x.

Ans. $x=12$ or -6.

13. Given $x^2+6x=27$, to find the values of x.

Ans. $x=3$, or -9.

14. Given $3x^2-2x=65$, to find the values of x.

Ans. $x=5$, or $-4\frac{1}{3}$.

15. Given $x^2-37x=-252$, to find the values of x.

Ans. $x=28$, or 9.

16. Given $x^2-10x+17=1$, to find the values of x.

Ans. $x=8$, or 2.

17. Given $7x^2-21x+13=293$, to find the values of x.

Ans. $x=8$, or -5.

18. Given $x^2-x-40=170$, to find the values of x.

Ans. $x=15$, or -14.

19. Given $x^2-26x+105=0$, to find the values of x.

Ans. $x=5$, or 21.

20. Given $11x^2-59x+78=0$, to find the values of x.

Ans. $x=$?

21. Given $9x^2-7x=116$, to find the values of x.

Ans. $x=4$, or $-3\frac{2}{9}$.

22. Given $9x^2-x=140$, to find the values of x.

Ans. $x=4$, or $-3\frac{8}{9}$.

23. Given $4x-\frac{36-x}{x}=46$, to find the values of x.

Ans. $x=12$ or $-\frac{3}{4}$.

24. Given $\frac{x+3}{2}+\frac{16-2x}{2x-5}=\frac{26}{5}$, to find the values of x.

Ans. $x=5$, or $\frac{69}{10}$.

25. Given $14+4x-\frac{x+7}{x-7}=3x+\frac{9+4x}{3}$, to find the values of x

Ans. $x=9$, or 28.

26. Given $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$, to find the values of x.

Ans. $x=21$, or 5.

27. Given $\frac{x}{x+60}=\frac{7}{3x-5}$, to find values of x.

Ans. $x=14$, or -10.

28. Given $\frac{8x}{x+2}-6=\frac{20}{3x}$, to find the values of x.

Ans. $x=10$, or $-\frac{2}{3}$.

29. Given $\frac{x+11}{x}+\frac{9+4x}{x^2}=7$, to find the values of x.

Ans. $x=3$, or $-\frac{1}{2}$.

30. Given $\frac{2x+9}{9}+\frac{4x-3}{4x+3}=\frac{3x+38}{18}$, to find the values of x.

Ans. $x=6$, or $-\frac{19}{4}$.

31. Given $\frac{x+12}{x}+\frac{x}{x+12}=\frac{78}{15}$, to find the values of x.

Ans. $x=3$, or -15.

32. Given $(\sqrt{4x+5})\times(\sqrt{7x+1})=30$, to find the values of x.

Ans. $x=5$, or $-\frac{179}{28}$.

33. Given $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x^2-x}{4}$, to find the values of x.

Ans. $x=4$, or 1, or $\frac{-3\pm\sqrt{-7}}{2}$.

34. Given $\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\frac{5}{11}$, to find the values of x.

Ans. $x=8$, or $-\frac{8}{9}$.

35. Given $\frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}=(x-2)^2$, to find the values of x.

Ans. $x=5$, or 3.

36. Given $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$, to find the values of x. *Ans.* $x=3$, or $-\frac{9}{2}$.*

* NOTE.—In this, and some other examples, only the real roots are given.

37. Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$, to find the values of x.
Ans. $x=4$, or 69.

38. Given $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$, to find the values of x.
Ans. $x=9$, or -12.

39. Given $(x+6)^2+2x^{\frac{1}{2}}(x+6)=138+x^{\frac{1}{2}}$, to find the values of x.
Ans. $x=4$, or 9.

40. Given $x-1=2+\frac{2}{x^{\frac{1}{2}}}$, to find the values of x.

Ans. $x=4$, or 1.

41. Given $x^4-2x^3+x=132$, to find the values of x.
Ans. $x=4$, or -3.

42. Given $x=\frac{12+8x^{\frac{1}{2}}}{x-5}$, to find the values of x.

Ans. $x=9$, or 4.

43. Given $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$, to find the values of x.

Ans. $x=2$, or $-\frac{8}{7}$.

44. Given $x-\frac{8}{\sqrt{x}}-\frac{2}{x}=5\left(1+\frac{2}{x}\right)$, to find the values of x.

Ans. 9, or 4.

45. Given $4x^4+\frac{x}{2}=4x^3+33$, to find the values of x.

46. Given $\sqrt[6]{\frac{1}{x^4}}+\sqrt[3]{\frac{1}{x}}=\frac{3-\sqrt[3]{x^2}}{x}$, to find the values of x.

Ans. $x=1$, or $-\frac{27}{8}$.

47. Given $x(\sqrt{x}+1)^2=102(x+\sqrt{x})-2576$, to find the values of x.
Ans. $x=49$, or 64.

48. Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$, to find the values of x.

Ans. $x=\pm a\left(\frac{1\pm\sqrt{5}}{2}\right)^{\frac{1}{2}}$.

49. Given $x^4-10x^3+35x^2-50x+24=0$, to find the values of x. *Ans.* $x=1$, 2, 3, or 4.

50. Given $2x^{\frac{3}{2}}(x^3+a^3)^{\frac{1}{2}}=2x^2(x+2a)+a^2(x-a)$ to find the values of x.

Ans. $x=\frac{a}{2}$, or $-a$.

(122.) In some examples, we may obtain values of the unknown quantity, which require to be taken with some limitation. If, in example 30, we substitute 9 in the place of x, the equation becomes $(9+6)^2+6(9+6)=138+3$, or $225+90=141$, which is not correct. But in solving this example it will be found that $x^{\frac{1}{2}}$ is equal to -3, and therefore in substituting 9 for x in the equation, we should place for $x^{\frac{1}{2}}$, -3 instead of $+3$. Observing this, the equation becomes by substituting, $(9+6)^2-6(9+6)=138-3$, or, $225-90=135$, a true result.

ADFECTED QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

(123.) An equation which contains two unknown quantities, and has terms in which the sum of the exponents of the unknown quantities equals, but never exceeds 2, is said to be of the second degree. Thus, $xy+x^2+y=a$, $xy+y^2+h=a$, are equations of the second degree.

(124.) If from two equations of the second degree, which contain two unknown quantities, and which are of the general form $ax^2+bxy+cy^2+dx+ey+f=0$, we eliminate one of the unknown quantities, we shall obtain an equation of the fourth degree, which contains only one of the unknown quantities. Hence, in general, the solution of two equations containing two unknown quantities gives rise to the solution of an equation of the fourth

degree. There are, however, many examples which may be solved by the aid of rules already given. No general rule can be given for the solution of such examples. The learner must depend on his own ingenuity.

EXAMPLES.

1. Given $\left\{\begin{array}{ll} y^4-432=12xy^2 & (1) \\ y^2=12+2xy & (2) \end{array}\right\}$ to find the values of x and y.

By transposing in (1) and (2) we have, $\left\{\begin{array}{ll} y^4-12xy^2=432 & (3) \\ y^2-2xy=12 & (4) \end{array}\right\}$

By completing the square in (3) and (4) we have, $\left\{\begin{array}{ll} y^4-12xy^2+36x^2=432+36x^2 & (5) \\ y^2-2xy+x^2=12+x^2 & (6) \end{array}\right\}$

Dividing eq. (5) by eq. (6), $\dfrac{y^4-12xy^2+36x^2}{y^2-2xy+x^2}=36$ (7)

Extracting sq. root of eq. (7), $\dfrac{y^2-6x}{y-x}=6$ (8)

Clearing (8) of fractions, $y^2-6x=6y-6x$; (9)

$\therefore \quad y^2=6y,$ (10)

and $\quad y=6$ (11)

Substitute this value of y in equation (2), and it becomes

$36=12+12x$ (12)

Whence, $x=2$

2. Given $\left\{\begin{array}{ll} x+y=6 & (1) \\ x^4+y^4=272 & (2) \end{array}\right\}$ to find the values of x and y.

$x^4+4x^3y+6x^2y^2+4xy^3+y^4=1296$ (3)

Eq. (3) is the 4th power of eq. (1)

Eq. (3)—eq. (2) gives $4x^3y+4xy^3+6x^2y^2=1024$ (4)

Eq. (1) squared gives $x^2+y^2+2xy=36$ (5)

By transposing, $x^2+y^2=36-2xy$ (6)

From (4), we have, $4xy(x^2+y^2)+6x^2y^2=1024$ (7)

For x^2+y^2 in (7), substitute its value, as found in (6), and we have,

$144xy-2x^2y^2=1024$ (8)

Whence, $x^2y^2-72xy=-512$ (9)

If we regard xy as the unknown quantity, equation (9) is a quadratic, and by Rule II., we find that,

$xy=8$, or 64 (10)

From the nature of the example we know that the second value should be rejected.

Eq. (5) is	$x^2+2xy+y^2=36$	(11)
Multiply eq. (10) by 4,	$4xy=32$	(12)
Eq. (11)—eq. (12) gives	$x^2-2xy+y^2=4$	(13)
Extracting sq. root	$x-y=2$	(14)
Eq. (1)+eq. (14) gives	$2x=8$	(15)
and,	$x=4$	(16)
$\therefore$	$y=2$	(17)

3. Given $\left\{\begin{array}{ll} x^4+y^4=1+2xy+3x^2y^2 & (1) \\ x^3+y^3=2y^2x+2y^2+x+1 & (2) \end{array}\right\}$ to find the values of x and y.

By transposing $2x^2y^2$ in eq. (1), we have,

$$x^4-2x^2y^2+y^4=1+2xy+x^2y^2, \quad (3)$$

Whence,
$$x^2-y^2=1+xy, \quad (4)$$

Or,
$$x^2-xy-y^2=1 \quad (5)$$

From (2) by transposing, $x^3+y^3-2y^2x-2y^2=x+1$ (6)

Let $x=ny$; then $x^2=n^2y^2$, and $x^3=n^3y^3$. In the place of x, x^2, and x^3, in equations (5) and (6), substitute their values, and we have

$$n^2y^2-ny^2-y^2=1, \quad (7)$$

and,
$$n^3y^3+y^3-2ny^3-2y^2=ny+1, \quad (8)$$

Eq. (8)$\div$eq. (7) gives
$$\frac{n^3y^3+y^3-2ny^3-2y^2}{n^2y^2-ny^2-y^2}=ny+1 \quad (9)$$

Divide the numerator and denominator of the first member of the last equation by y^2, and then clear the equation of fractions, and we have,

$$n^3y+y-2ny-2=n^3y+n^2-n^2y-n-ny-1 \quad (10)$$

By transposing, $n^2y-ny+y=n^2-n+1$, (11)

Or, by factoring, $y(n^2-n+1)=n^2-n+1$; (12)

$$\therefore \quad y=1. \quad (13)$$

By substituting this value of y in equation (7), it becomes

$$n^2-n-1=1, \quad (14)$$

or,
$$n^2-n=2 \quad (15)$$

By Rule I.,
$$n=2; \quad (16)$$

$$\therefore \quad x=ny=2 \quad (17)$$

4. Given $\left\{\begin{array}{ll} x^2+xy=12 & (1) \\ xy-2y^2=1 & (2) \end{array}\right\}$ to find the values of x and y.

In this example each of the literal quantities is of the second degree; that is, the proposed quadratics are *homogeneous*. Such examples may always be reduced by substituting for one of the unknown quantities an unknown multiple of the other; because by this substitution we shall introduce the square of this other into every term, and therefore it may be eliminated from the two equations, and the resulting equation will be a quadratic involving the unknown multiple, and this being found, the values of the two unknown quantities are readily obtained.

Therefore, let $x=ny$; then, $x^2=n^2y^2$. Substitute for x and x^2 in the given equations, their values, and they become

$$n^2y^2+ny^2=12. \quad (3)$$

$$ny^2-2y^2=1. \quad (4)$$

From (3) we have, $$y^2=\frac{12}{n^2+n}. \quad (5)$$

From (4) we have, $$y^2=\frac{1}{n-2}; \quad (6)$$

$$\therefore \quad \frac{12}{n^2+n}=\frac{1}{n-2}. \quad (7)$$

Clearing (7) of fractions, $n^2+n=12n-24$, (8)

or, $n^2-11n=-24$ (9)

By Rule I., $n=3$, or 8; (10)

$\therefore \quad n^2=9$, or 64. (11)

By substituting these values of n and n^2 in equation (3), we have,

$$9y^2+3y^2=12, \quad (12)$$

and, $$64y^2+8y^2=12; \quad (13)$$

Whence, $$y=\pm 1, \text{ or } \pm\frac{1}{\sqrt{6}},$$

and, $$x=ny=\pm 3, \text{ or } \pm\frac{3}{\sqrt{6}}.$$

5. Given $\left\{\begin{array}{ll} \dfrac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}=\dfrac{17}{4}-\dfrac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}} & (1) \\ x(x+y)=52-\sqrt{x^2+xy+4} & (2) \end{array}\right\}$, to find the values of x and y.

Let $n=\dfrac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}$; then $\dfrac{1}{n}=\dfrac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}$. Substitute, in equation (1), n and $\dfrac{1}{n}$ in the place of their values, and it becomes

$$n+\frac{1}{n}=\frac{17}{4}, \qquad (3)$$

or, $$4n^2-17n=-4; \qquad (4) \text{ from } (3)$$

$\therefore$ by Rule I., $$n=4, \text{ or } \tfrac{1}{4}; \qquad (5)$$

Whence, $$\frac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}=4, \text{ or } \tfrac{1}{4}. \qquad (6)$$

Clearing (6) of fractions,

$$x+\sqrt{x^2-y^2}=4x-4\sqrt{x^2-y^2}, \qquad (7)$$

or, $$x+\sqrt{x^2-y^2}=\frac{x-\sqrt{x^2-y^2}}{4}. \qquad (8)$$

By transposing in (7), $5\sqrt{x^2-y^2}=3x.$ (9)

By squaring eq. (9), $25x^2-25y^2=9x^2,$ (10)

or, $16x^2=25y^2$ (11)

Whence, $4x=5y$, and $x=\dfrac{5y}{4}$ (12)

We shall obtain the same result if we employ equation (8). By adding 4 to each member of equation (2), and transposing, it may be written

$$x^2+xy+4+\sqrt{x^2+xy+4}=56 \qquad (13)$$

Let $s=\sqrt{x^2+xy+4}$; then $x^2+xy+4=s^2$. Substitute s^2 and s for their values, and equation (13) becomes

$$s^2+s=56, \qquad (14)$$

$$\therefore \quad s=\pm\frac{15}{2}-\frac{1}{2}=7 \text{ or } -8 \qquad (15)$$

and $$s^2=49, \text{ or } 64 \qquad (16)$$

$$\therefore \quad x^2+xy+4=s^2=49, \text{ or } 64. \qquad (17)$$

By substituting in the place of x, in equation (17), its value as found in (12), it becomes, by reduction,

$$y=\pm 4, \text{ or } \pm\frac{8}{\sqrt{3}};$$

and $$x=\frac{5y}{4}=\pm 5, \text{ or } \pm\frac{10}{\sqrt{3}}.$$

6. Given $\left\{\begin{array}{ll} x^2+3x+y=73-2xy. & (1) \\ y^2+3y+x=44. & (2) \end{array}\right\}$ to find the values of x and y.

By transposing in (1), $\quad x^2+2xy+3x+y=73. \quad (3)$
Eq. (2)+eq. (3) $\quad x^2+2xy+y^2+4x+4y=117, \quad (4)$
or, $\quad (x+y)^2+4(x+y)=117. \quad (5)$

If we regard $x+y$ as the unknown quantity, equation (5) is a quadratic, and by Rule II., we find that

$$x+y=\pm 11-2=9 \text{ or } -13 \quad (6)$$
$$\therefore\ x=9-y, \text{ or } -13-y \quad (7)$$

By substituting these values of y in equation (2), we have

$y^2+2y=35, \quad (8)$
And, $\quad y^2+2y=57. \quad (9)$
From (8), $\quad y=5, \text{ or } -7 \quad (10)$
From (9), $\quad y=-1+\sqrt{58}, \text{ or } -1-\sqrt{58} \quad (11)$
Hence, $\quad x=9-y=4, \text{ or } 16, \quad (12)$
And, $\quad x=-13-y=-12-\sqrt{58}, \text{ or } -12+\sqrt{58} \quad (13)$

The values of x in (12) correspond to the values of y in (10) and the values of x in (13), to the values of y in (11).

7. Given $\left\{\begin{array}{ll} x^2+x+y=18-y^2 & (1) \\ 2xy=\cancel{6}2 & (2) \end{array}\right\}$ to find the values of x and y. *Ans.* $x=3$, $y=2$.

8. Given $\left\{\begin{array}{ll} x^2+2xy+y^2+2x+2y=120 & (1) \\ xy-y^2=8 & (2) \end{array}\right\}$ to find the values of x and y. *Ans.* $x=6$, or 9, $y=4$, or 1.

9. Given $\left\{\begin{aligned} x^2+y^2-x-y&=78 \quad (1)\\ xy+x+y&=39 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=9$, or 3, $y=3$, or 9.*

10. Given $\left\{\begin{aligned} x^2y^4-7xy^2&=1710 \quad (1)\\ xy-y&=12 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=5$, or $\frac{1}{5}$, $y=3$, or -15.

11. Given $\left\{\begin{aligned} xy+xy^2&=12 \quad (1)\\ y+xy^3&=18 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $y=2$, or $\frac{1}{2}$, $x=2$, or 16.

12. Given $\left\{\begin{aligned} x^2+2x^3y&=441-x^4y^2 \quad (1)\\ xy&=3+x \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=3$, or -7, $y=2$, or $\frac{4}{7}$.

13. Given $\left\{\begin{aligned} x^2+4y^2&=256-4xy \quad (1)\\ 3y^2-x^2&=39 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=\pm 6$, or ± 102, $y=\pm 5$, or ± 59.

14. Given $\left\{\begin{aligned} 4xy&=96-x^2y^2 \quad (1)\\ x+y&=6 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=4$, or 2, $y=2$, or 4.

15. Given $\left\{\begin{aligned} (x^2+1)y&=xy+126 \quad (1)\\ (x^2+1)y&=x^2y^2-744 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=5$, or $\frac{1}{5}$, $y=6$, or 150.

16. Given $\left\{\begin{aligned} x+y+\sqrt{x+y}&=12 \quad (1)\\ x^3+y^3&=189 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=5$, or 4, $y=4$, or 5.

17. Given $\left\{\begin{aligned} x^2+y^2+x-y&=132 \quad (1)\\ (x^2+y^2).(x-y)&=1220 \quad (2)\end{aligned}\right\}$ to find the values of x and y.
Ans. $x=11$, or -1, $y=1$, or -11.

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* NOTE.—In this and some of the following examples, the rational roots only are given.

18. Given $\left.\begin{cases} x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x & (1) \\ x^{\frac{1}{2}}+y^{\frac{1}{3}}=x & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=4$, or 1, $y=8$.

19. Given $\left.\begin{cases} x+x^{\frac{1}{2}}=\dfrac{y^2+y+2}{x^{\frac{1}{2}}}+4 & (1) \\ y+xy=y^2+4y & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=4$, or 1, $y=1$, or -2.

20. Given $\left.\begin{cases} \dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}=\dfrac{16}{15}+\dfrac{2x-\sqrt{y}}{2x+\sqrt{y}} & (1) \\ 2x+y\ =26-7\sqrt{2x+y+4} & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=2$, or -10, $y=1$, or 25.

21. Given $\left.\begin{cases} x+y=5 & (1) \\ (x^2+y^2).(x^3+y^3)=455 & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=3$, or 2; $y=2$, or 3.

22. Given $\left.\begin{cases} x^4+y^4=97 & (1) \\ x+y=5 & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=3$, or 2, $y=2$, or 3.

23. Given $\left.\begin{cases} (2x-4y)^2+x-2y=5 & (1) \\ x^2-y^2=8 & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=3$, or $\frac{17}{3}$, $y=1$, or $\frac{7}{3}$.

24. Given $\left.\begin{cases} xy=24+x+y & (1) \\ \sqrt{9+x^2+y^2}=\frac{1}{8}(x^2+y^2) & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=6$.

25. Given $\left.\begin{cases} 2xy-24(x+y)=-240 & (1) \\ x^2+y^2=100 & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=8$, or 6, $y=6$, or 8.

26. Given $\left.\begin{cases} x^2+y^2=29 & (1) \\ x^3y+y^3x=290 & (2) \end{cases}\right\}$ to find the values of x and y.

Ans. $x=5$, $y=2$.

27. Given $\left\{\begin{array}{ll}\left(\dfrac{3x-2y}{2x}\right)^{\frac{1}{2}}+\left(\dfrac{2x}{3x-2y}\right)^{\frac{1}{2}}=2 & (1)\\ x^2-18=4xy-9x & (2)\end{array}\right\}$
to find the values of x and y. *Ans.* $x=6$, $y=3$.

28. Given $\left\{\begin{array}{ll}x+y=6 & (1)\\ x^5+y^5=1056 & (2)\end{array}\right\}$ to find the values of x and y. *Ans.* $x=2$, or 4, $y=4$, or 2.

29. Given $\left\{\begin{array}{ll}\dfrac{x+\sqrt{x+y}}{x-\sqrt{x+y}}+\dfrac{-\sqrt{x}+x+y}{\sqrt{x}+x+y}=\dfrac{89}{40} & (1)\\ y^2-\sqrt{xy^2}=\dfrac{4x}{9} & (2)\end{array}\right\}$ to find the values of x and y. *Ans.* $x=9$, $y=4$.

30. Given $\left\{\begin{array}{ll}\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}=\dfrac{24}{5} & (1)\\ \left(\dfrac{x-y}{x^4}\right)^{\frac{1}{2}}+\dfrac{1}{x}=\dfrac{4}{9\sqrt{x-y}} & (2)\end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l}x=3, \text{ or } \frac{45}{2}; \text{ or } \frac{3}{16}, \text{ or } \frac{45}{32},\\ y=2, \text{ or } -\frac{135}{4}; \text{ or } \frac{1}{8}, \text{ or } -\frac{135}{64}.\end{array}\right.$

PROBLEMS IN ADFECTED QUADRATICS.

(125.) If a proper notation be employed, some of the following examples are susceptible of solutions which do not involve adfected quadratic equations. The student should endeavor to obtain different solutions of the same problem by adopting different notations, and thus strengthen his powers of analysis.

1. There are two numbers, one of which is greater than the other by 8, and whose product is 240. What are the numbers?

First Solution.

Let $x=$ the less number;*

Then, $x+8=$ the greater number.

By the question, $x\times(x+8)=240$,

Or, $x^2+8x=240$;

Whence, by Rule II., $x=-4\pm\sqrt{256}=12$, or -20,

And, $x+8=20$, or -12; hence, the numbers are 12 and 20, or -12 and -20.

Second Solution.

Let $x+4=$ the greater;

Then, $x-4=$ the less.

$\therefore\quad (x+4)\times(x-4)=240$,

Or, $x^2-16=240$;

$\therefore\quad x^2=256$,

And $x=\pm 16$;

Whence, $x+4=20$, or -12, $x-4=12$, or -20.

2. A person dies, leaving children, and a fortune of \$48600, which, by the will, is to be divided equally amongst them. It happens, however, that immediately after the death of the father, two of the children also die. If, in consequence of this, each remaining child received \$1950 more than he or she was entitled to by the will, how many children were there?

Let $x+1=$ the whole number of children;

Then $x-1=$ the number after two had died.

Put $a=1950$

Then $24a=48600=24\times1950$.

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* NOTE.—The greater of two numbers is equal to half their sum increased by half their difference, and the less is equal to half their sum diminished by half their difference. For, let x the greater, and y the less number; and let $2s=$ their sum, and $2d=$ their difference. Then

$$x+y=2s,$$

$$\text{And } x-y=2d;$$

Whence, $x=s+d$, and $y=s-d$. Q. E. D.

Now, $\frac{24a}{x+1}$ = what each child would have rec'd by the will.

And, $\frac{24a}{x-1}$ = " " " did receive.

By the question, $\frac{24a}{x+1}+a=\frac{24a}{x-1}$

Clearing of fractions, $24ax-24a+ax^2-a=24ax+24a$

By transposing, $ax^2=49a$,

or, $x^2=49$

$\therefore \quad x=7$,

and, $x+1=8$, the number of children.

If we had represented the number of children by x, instead of $x-1$, the solution of the problem would have involved an adfected quadratic equation.

3. A surveyor is requested to lay out a piece of land in a rectangular form, so that its perimeter may be 100 rods, and its area, 589 square rods. What is its length, and width ?

Since its perimeter is 100 rods, its length and width, together, must $=$ 50 rods.

Let x = the length ;

Then $50-x$ = the width.

By the question, $x\times(50-x)=589$,*

Or, $x^2-50x=-589$

By Rule II., $x=+25\pm\sqrt{625-589}=31$, or 19

$\therefore \; 50-x=19$, or 31.

Hence, the length is 31 rods, and the width 19.

4. A set out from C towards D, and travelled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day $\frac{1}{19}$th of the whole journey ; and after he had travelled as many days as he went miles in one day, he met A. Required the distance from C to D.

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* Note.—The area of any rectangle is equal to the product of its length and width.

Let $19x$=the required distance;
Then, x=the number of miles B travelled per day,
And, x= " " of days he travelled before he met A.
∴ $7x+32$= " " of miles A travelled,
And, x^2= " " " B "

Whence, $x^2+7x+32=19x$,

Or, $x^2-12x=-32$

By Rule II., $x=6\pm\sqrt{36-32}=8$, or 4.

∴ $19x=152$, or 76. Both of these values will answer the conditions of the question. The distance, then, of C from D was 152, or 76 miles.

5. A grazier bought as many lambs as cost him \$60; out of which he reserved 15, and sold the remainder for \$54, gaining 10 cents a head by them. How many lambs did he buy, and what was the price of each?

Let x=the number,

Then, $\frac{60}{x}$=the price of each.

By the question, $(x-15)\times\left(\frac{60}{x}+\frac{1}{10}\right)=54$ (1)

Or $(x-15)\times(600+x)=540x$ (2)

And $x^2+585x-9000=540x$ (3)

By transposing, $x^2+45x=9000$ (4)

By Rule I., $x=\frac{-45\pm\sqrt{9(4000+225)}}{2}=\frac{-45\pm3\sqrt{4225}}{2}$

Or, $x=\frac{-45\pm3\times65}{2}=75$, or -120

From the nature of the problem, we know that the negative value of x should be rejected.

$\frac{60}{75}=0.80$. Therefore he bought 75 lambs, and paid 80 cents a head.

6. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two thirds of the time that Silenus would have taken to empty

the whole cask. After that Silenus awoke, and drank what Bacchus had left. Had they drank both together, it would have been emptied two hours sooner, and Bacchus would have drank only half what he left Silenus. Required the time in which they would empty the cask separately.

Let x= the number of hours, in which Bacchus would drink it,
And y= " " " Silenus " "

Since Bacchus would empty the cask in x hours, in 1 hour he would empty $\frac{1}{x}$th part of it, and by the same reasoning, Silenus would drink $\frac{1}{y}$th part of it in 1 hour; hence, both together would drink, in 1 hour, $\frac{1}{x}$th part plus $\frac{1}{y}$th part of the cask, or a part denoted by $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$. Now, as the whole cask may be represented by 1, the time in which both would have emptied the cask, if they had drank together, may be found, by dividing unity by the part of the cask which they both drink in 1 hour. Hence $1\div\frac{x+y}{xy}=\frac{xy}{x+y}$= the time in which both would have drank the cask, by drinking together. By the question $\frac{1}{x}\times\frac{2y}{3}=\frac{2y}{3x}$= the part which Bacchus drinks. Therefore, $1-\frac{2y}{3x}=\frac{3x-2y}{3x}$= the part which was left for Silenus to drink. Therefore, $\frac{3x-2y}{3x}\div\frac{1}{y}=\frac{3xy-2y^2}{3x}$= the time which Silenus required to drink his part.

Again, by the question, $\frac{3x-2y}{3}\times\frac{1}{2}=\frac{3x-2y}{6x}$= the part which Bacchus would have drank, had they both drank together. The time which Bacchus occupied in drinking this quantity may be found by dividing the whole quantity by the quantity which he

drank in one hour. Therefore, $\frac{3x-2y}{6x}\div\frac{1}{x}=\frac{3x-2y}{6}=$ the time that Bacchus would have drank if they had both drank together, or the time in which both, by drinking together, would have drank it. Therefore,

$$\frac{3x-2y}{6}=\frac{xy}{x+y}. \qquad (1)$$

By another condition of the question, we have,

$$\frac{2y}{3}+\frac{3xy-2y^2}{3x}-2=\frac{xy}{x+y}, \qquad (2)$$

Clearing (1) of fractions, $3x^2+xy-2y^2=6xy$. (3)

Let $x=ny$. Substitute this value of x in equation (3), and it becomes

$$3n^2y^2+ny^2-2y^2=6ny^2. \qquad (4)$$

Dividing (4) by y^2, $3n^2+n-2=6n$

By transposing, $3n^2-5n=2$;

Whence, $n=2$,

and, $x=ny=2y$

Substitute $2y$ for x in equation (2), and it becomes,

$$\frac{2y}{3}+\frac{6y^2-2y^2}{6y}-2=\frac{2y^2}{3y}.$$

By reducing, $y=3$

$\therefore \quad x=2y=6$

7. What two numbers are those whose sum is 19, and whose difference multiplied by the greater is 60?

Ans. 12 and 7.

8. There is a field in the form of a rectangular parallelogram, whose length exceeds its breadth by 16 yards; and it contains 960. Required the length and breadth.

Ans. 40, and 24 yards.

9. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the journey. How many miles did he go per hour?

Ans. 7 miles.

10. From two places, at the distance of 320 miles, two persons, A and B, set out at the same time to meet each other. A travelled 8 miles a day more than B, and the number of days in which they met was equal to half the number of miles B went in a day. How many miles did each travel per day, and how far did each travel?

Ans. A went 24, B 16 miles per day; A went 192, and B 128 miles.

11. A tailor bought a piece of cloth for \$147, from which he cut off 12 yards for his own use, and sold the remainder for \$120.25, gaining 25 cents per yard. How many yards were there, and what did it cost him per yard?

Ans. 49 yards, at \$3 per yard.

12. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has 4 men more in a side than the other. What number of men are in a side of each of the squares?

Ans. 21, and 25.

13. What number is that, to which if 24 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18. *Ans.* 25.

14. A poulterer bought 15 ducks, and 12 turkeys for 105 shillings. He had 2 ducks more for 18 shillings than he had of turkeys for 20 shillings. What was the price of each?

Ans. The price of a duck was 3*s.*, and of a turkey 5*s.*

15. The joint stock of two partners, A and B, was \$416. A's money was in trade 9 months, and B's 6 months; when they shared stock and gain. A received \$228, and B \$252. What was each man's stock? *Ans.* A's 192, and B's \$224.

16. Three merchants, A, B, and C, made a joint stock, by which they gained a sum less than that stock by \$80. A's share of the gain was \$60; and his contribution to the stock was \$17 more than B's. Also B and C together contribute \$325. How much did each contribute?

Ans. A \$75, B \$58, C \$267.

17. A and B hired a pasture into which A put 4 horses, and B as many as cost him 18 shillings a week. Afterwards B put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

Ans. 30 shillings per week.

18. A and B engage to reap a field for £4 10*s.*; and as A alone could reap it in 9 days, they promise to complete it in 5 days. They found, however, that they were obliged to call in C, an inferior workman, to assist them for the two last days, in consequence of which B received 3*s.* 9*d.* less than he otherwise would have done. In what time could B or C alone reap the field?

Ans. B in 15, C in 18 days.

19. What two numbers are they whose product is 255, and the sum of whose squares is 514? *Ans.* 15 and 17.

20. What two numbers are they, whose difference is 8, and the sum of whose squares is 544?

Ans. 12 and 20.

21. What two numbers are they, whose sum is 41, and the sum of whose squares is 901? *Ans.* 15 and 26.

22. Divide the number 16 into two such parts, so that the products of the two parts added to the sum of their squares may be 208. *Ans.* The parts are 4 and 12.

23. Find two numbers, whose difference, multiplied by the difference of their squares$=160$; and whose sum, multiplied by the sum of their squares$=580$. *Ans.* 3 and 7.

24. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only 4 revolutions more than the hind wheel in going the same space. Required the circumference of each?

Ans. 5 and 4 yards.

25. Find two numbers such, that their sum and product together may equal 34, and the sum of their squares exceed the sum of the numbers themselves by 42. What numbers are they?

Ans. 4 and 6.

26. A and B were going to market, the first with cucumbers and the second with three times as many eggs; and they find that if B gave all his eggs for the cucumbers, A would lose 10 cents, according to the rate at which they were then selling. A therefore reserves two fifths of his cucumbers; by which B would lose 6 cents, according to the same rate. But B, selling the cucumbers at 6 cents apiece, gains upon the whole the price of six eggs. Required the number of eggs and cucumbers, and their price.

Ans. 30 eggs and 10 cucumbers, and the price of an egg 1 cent, cucumber 4 cents.

27. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each?

Ans. £25, and £16.

28. A cask, whose contents is 20 gallons, is filled with wine, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture, and it appears that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of wine in each. Required the quantity of wine first drawn off. *Ans.* 10 gallons.

29. It is required to find two numbers such, that their sum, product, and difference of their squares, shall all be equal.

Ans. $\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$, and $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

30. A person buys a horse, and pays a certain sum for it; he then sells it for $144, and gains as much per cent. as the horse cost him. How much did the horse cost him?

Ans. $80.

31. A merchant buys a certain quantity of cloth, and pays a certain sum for it, besides 4 per cent. for carriage. He sells it for \$390, and gains as much per centum as the twelfth part of the purchase money amounted to. What did he buy it for?

Ans. \$300.

32. A rectangular garden, which is 8 rods long and 6 rods wide, is to be surrounded with a gravel walk of uniform width, so that the number of square rods in the walk may be equal to one half the whole number of square rods in the garden. What must be the width of the walk? *Ans.* 1 rod.

33. In the preceding problem, if the walk had extended half way around the garden, what would have been its width?

Ans. 2 rods.

34. A carpenter has a square board, a side of which is a inches, and he wishes to form a regular octagon out of the square, by cutting off from its four corners four equal right-angled triangles. What is the length of a side of one of the triangles, which is adjacent to the right-angle?

Ans. $a(1-\frac{1}{2}\sqrt{2})=a\times 0.293$.

35. The area of a rectangular garden is 560 square rods, and its length exceeds its width by 8 rods. Required its length and width. *Ans.* 28, and 20 rods.

36. Divide 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

37. A poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first. The third contained three times as many as the first and second; and the fourth contained 6 more than the square of one third the number in the third; and the whole number was 1938. How many were there in each flock?

Ans. The numbers were 18, 24, 126, and 1770.

38. There are two square buildings, that are paved with stones a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately?

Ans. 26 and 38 feet.

DISCUSSION OF AN EQUATION OF THE SECOND DEGREE.

(126.) We have seen that all quadratic equations may be reduced to the form

$$x^2+2ax=c; \qquad \text{(A)}$$

Whence,
$$x=-a\pm\sqrt{a^2+c}$$

By using the positive sign before the radical, and then the negative sign, we find that the two values of x are,

$$x=-a+\sqrt{a^2+c}, \qquad (1)$$

and,
$$x=-a-\sqrt{a^2+c}. \qquad (2)$$

Now, whether a is positive or negative, its square must be positive, and therefore the quantity under the radical will be *positive* when $a^2>c$, and *negative* when $a^2<c$, and c is negative. We have then two distinct cases.

CASE I.

When a^2 is greater than c.

In this case, if c is positive, the value of $\sqrt{a^2+c}$ must be *greater* than a, and therefore the first value of x is real and positive, whether a is positive or negative, and the second value of x is negative. But if c is negative, the value of $\sqrt{a^2+c}$ is *less* than a, and therefore the first value of x is negative, if a is positive, and positive if a is negative. The second value of x is positive when a is negative, and negative when a is positive. We conclude, therefore, that in the first case both values of x are real.

CASE II.

When a^2 is less than c,

If c is positive, we know, from what has been said in Case I., that the first value of x is positive, and that the second value of x

is negative. But, when c is negative, a^2+c must be a negative quantity, and therefore $\sqrt{a^2+c}$ is an imaginary quantity. Hence, we conclude, that in the second case, both values of x are imaginary when c is negative. We will show in a subsequent article, that when both values of the unknown quantity are imaginary, there must be conditions in the problem which gave rise to the equation that yielded these imaginary values, which cannot be fulfilled.

(127.) If we designate the two values of x by m and n, we have, from equations (1) and (2),

$$m=-a+\sqrt{a^2+c}, \qquad (3)$$

$$\text{and, } n=-a-\sqrt{a^2+c}. \qquad (4)$$

By adding equation (3) to equation (4), and then multiplying the one by the other, we have,

$$m+n=-2a, \qquad (5)$$

$$\text{and, } mn=-c \qquad (6)$$

Hence,

In any quadratic equation the sum of the two roots is equal to once the co-efficient of the first power of the unknown quantity taken with a contrary sign, and their product is equal to the absolute term taken with a contrary sign.

(128.) We will now show that c cannot be greater than a^2. To prove this, we must first establish the following

LEMMA.

If any given number be separated into two parts, the product of the two parts will be the greatest possible when the parts are equal.

For, let $2a=$the given number,
and $2x=$the difference of the two parts.
Then, $a+x=$the greater part,
and, $a-x=$the less part.
Let $P=$their product,
Then, $(a+x).(a-x)=P$,
or, $a^2-x^2=P$

Now, it is obvious that as x diminishes P must increase, and therefore the product will be the greatest possible when $x=0$. But if $x=0$, $a+x$ and $a-x$ are each equal to a. Hence, the Lemma is true.

(**129.**) Since, in equation (A), $2a$ is the sum of the two roots, and c their product, it follows, from the preceding *Lemma*, that c cannot be greater than a^2. Now, when c is negative, and greater than a^2, each of the values of the unknown quantity is imaginary. We therefore conclude that when the solution of a problem gives rise to a quadratic equation, the roots of which are imaginary, that problem must contain conditions which cannot be fulfilled.

(**130.**) If we represent the two roots of equation (A) by m and n, as before, we know from what has been shown that the equation may be written,

$$x^2-(m+n)x=-mn,$$

Or, by transposing, $x^2-(m+n)x+mn=0$,

Or, by factoring, $(x-m)\times(x-n)=0$.

Hence,

Every quadratic equation may be resolved into two factors, one of which is the first power of the unknown quantity connected with one of the roots with its sign changed, and the other is the first power of the unknown quantity connected with the other root with its sign also changed.

(**131.**) We will now illustrate the principles which have been established in the preceding discussion, by applying them to the solution of problems.

1. It is required to divide the number 24 into two parts, such that their product shall be 150.

Let $x=$ one of the parts,

Then $24-x=$ the other part.

By the question, $(24-x)\times x=150$,

Or, $24x-x^2=150$,

Or, $x^2-24x=-150$,

Whence, by Rule II., $x=12\pm\sqrt{144-150}$,

Or, $x=12\pm\sqrt{-6}$.

Since the two values of x are imaginary, we know that it is *impossible* to divide the number 24 into two such parts, that their product shall be 150.

2. To find on a line, A B, which joins two lights of different intensities, a point which is equally illuminated by each, admitting that it is a law of light that the intensities of the same light at different distances are inversely as the squares of the distances.

$$P_3 \qquad\qquad A \qquad\qquad P_1 \qquad\quad B \qquad\qquad P_2$$

Let a = the distance A B between the two lights,

b = the intensity of the light A at the distance of 1 ft. from A,

And c = the intensity of the light B at the distance of 1 ft. from B.

Let P_1 be the point required.

Let x = the distance, A P_1. $\therefore \; a-x = P_1 B$.

By the law which we have admitted, since the intensity of the light A at the distance of 1 foot is b, its intensity at the distance of x feet must be $\frac{b}{x^2}$; and the intensity of the light B at the distance of $a-x$ feet must be $\frac{c}{(a-x)^2}$. But, by the question, these two intensities are equal. Therefore, we have

$$\frac{b}{x^2} = \frac{c}{(a-x)^2} \qquad (1)$$

$$\text{Or,} \quad \frac{b(a-x)^2}{x^2} = c \qquad (2)$$

$$\text{Or,} \quad \frac{(a-x)^2}{x^2} = \frac{c}{b} \qquad (3)$$

$$\text{By extracting the square root,} \quad \frac{a-x}{x} = \frac{\pm\sqrt{c}}{\sqrt{b}} \qquad (4)$$

By reducing the last equation, we find that the two values of x are

$$\left\{\begin{array}{ll} x = \dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}} & (1) \\[2ex] \text{and, } x = \dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}} & (2) \end{array}\right\} \text{ Whence } \left\{\begin{array}{l} a-x = \dfrac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}} \\[2ex] a-x = \dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}} \end{array}\right\}$$

Now, we may have five cases in the discussion of this problem.

1. *When b is greater than c.*
2. *When b is less than c.*
3. *When $b=c$.*
4. *When $b=c$, and $a=0$.*
5. *When $a=0$, and b and c are not equal.*

CASE I.

When $b>c$.

The first value of x is positive and less than a, since a is multiplied by a proper fraction, $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$. Hence, this value of x shows that there is a point P_1, between A and B, which is equally illuminated by the two lights. And this point must be nearer to B than to A. For, since $b>c$, $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$, and therefore $a\times\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, or $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$. Now, this is the conclusion to which we ought to arrive, since we here suppose that the intensity of the light A is greater than that of B. The corresponding value of $a-x$ is positive and less than $\frac{a}{2}$.

The second value of x is positive and *greater* than a, since a is multiplied by an improper fraction, $\frac{\sqrt{b}}{\sqrt{b}-c}$. Hence, this value of x shows that there is a point P_2, in the prolongation of A B, and to the right of the light B, which is equally illuminated by the two lights. In fact, since the two lights emit rays in all directions, there may be a point in the prolongation of A B which is equally illuminated by the two lights. But this point must be to the right of the light B, so that it may be nearer to the weaker light.

We may readily perceive why these two values are derived from the same equation. If we had taken $A P_2$ for the unknown quantity, instead of $A P_1$, the equation would have been

$$\frac{b}{x^2}=\frac{c}{(x-a)^2}.$$

Now, as $(a-x)^2$ is identical with $(x-a)^2$, the new equation is the same as the first equation, the solution of which ought, therefore, to give $A P_2$ as well as $A P_1$.

The second value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it should be, since $x>a$. By changing the signs of each member of the equation $a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, it becomes $x-a=\frac{a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, and this value of $x-a$ expresses the real length of $B P_2$.

CASE II.

When $b<c$.

In this case we obviously have,

$$\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}<\frac{1}{2},$$

And, $$\frac{\sqrt{c}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}, \text{ and } <1.$$

Therefore, $$\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}<\frac{a}{2},$$

And $$\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}, \text{ and } <a.$$

Hence, we conclude that *the first value of x* is less than one half of a, and therefore the point of equal illumination is between A and B, and nearer to A than B, as it should be, since by the present hypothesis, the weaker light is A.

Since $\sqrt{b}-\sqrt{c}$ is a negative quantity, c being greater than b, it follows that the second value of x is negative. Now, as the distances which have been measured from A towards the right have been regarded as positive, distances measured towards the left must be regarded as negative. Hence, the second value of x gives P_3, to the left of A, for the point of equal illumination. The corresponding value of $a-x$ is positive, for the reason that, x being negative, $a-x$ expresses the *sum* of two quantities, that is, $a-x$ is the sum of the two distances, $A\,B$ and $A\,P_3$, or the numerical value of the whole line $B\,P_3$.

CASE III.

When $b=c$.

In this case, the first values of x and $a-x$ are equal to each other, each reducing to $\frac{a}{2}$. Therefore the point of equal illumination is equally distant from the two lights, as it should be, since their intensities are equal.

The second values of x and $a-x$ are reduced to $\frac{a\sqrt{b}}{0}=\infty$; that is, the second point of equal illumination is at an *infinite* distance from A and B.

CASE IV.

When $b=c$, *and* $a=0$.

In this case, the first values of x and $x-a$ become 0, which shows that one of the points of equal illumination is the point where the two lights are situated.

The second values of x and $x-a$ become $\frac{0}{0}$, which is the symbol of an indeterminate quantity. (Art. 83.) This result shows that any point in the line $A\,B$ is equally illuminated by the two lights. In fact, we ought to arrive at this conclusion, since, by the present hypothesis, the intensities of the two lights are equal,

and they are situated at the same point; and, therefore, *every* point in the line $A\,B$ is illuminated by one of the lights as much as by the other.

CASE V.

When $a=0$, *and* b *is greater or less than* c.

In this case, each of the values of x and $a-x$ reduce to 0. Hence, we conclude that there is only one point which is equally illuminated by the two lights, and that this point is the one in which the two lights are placed.

CHAPTER VIII.

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

(132.) *An Arithmetical Progression is any series of quantities which increase or decrease by the addition or subtraction of the same quantity.*

Thus the numbers

$$1,\ 4,\ 7,\ 10,\ 13,\ \&c.,$$

each of which is obtained by adding 3 to the preceding term in the series, is an *increasing* Arithmetical Progression; and the numbers

$$13,\ 11,\ 9,\ 7,\ 5,\ 3,\ 1,$$

each of which is obtained by subtracting 2 from the preceding term, is a *decreasing* Arithmetical Progression.

(133.) In an *arithmetical progression* there are five quantities to be considered, namely, the *first term*, the *common difference*, the *number of terms*, the *last term*, and the *sum of all the terms*. Any three of these being given, the other two may be found. We shall establish two *fundamental formulas*, from which all the other formulas in arithmetical progression may be derived.

PROBLEM I.

(134.) *In an arithmetical progression, having given the first term, the common difference, and the number of terms, it is required to find the last term.*

Let a=the first term,
d=the common difference,
n=the number of terms,
And l=the last term.

Hence, the series will be,

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \text{ \&c.}$$

By examining this series, we find that the co-efficient of d in the *second term* is 1, in the *third term*, 2, in the *fourth term*, 3, that is, in each term, the co-efficient of d is *one less* than the number which marks the place of that term. Therefore, the nth, or last term must be $a+(n-1)d$. Hence, we have the equation

$$l=a+(n-1)d \qquad \text{(A)}$$

Formula (A) may be employed for finding the last term of a decreasing series, by regarding d as being negative.

PROBLEM II.

(135.) *Having given the first term, the last term, and the number of terms, it is required to find the sum of all the terms.*

Represent the sum of the series $a, a+d, a+2d, a+3d$ - - - - - l by S; then we have the equation,

$$S=a+(a+d)+(a+2d)+(a+3d) \text{ - - - - - } l \qquad (1)$$

$$\text{Or, } S=l+(l-d)+(l-2d)+(l-3d) \text{ - - - - - } a \qquad (2)$$

by writing the *same* series in a reverse order.

By adding equations (1) and (2) we have,

$$2S=(a+l)+(a+l)+(a+l) \text{ - - - - - } (a+l) \qquad (3)$$

Since there are n terms, and each term is $a+l$, equation (3) may be written

$$2S=(a+l)\times n$$

$$\therefore \quad S=\frac{a+l}{2}\times n \qquad \text{(B)}$$

(136.) By the aid of formulas (A) and (B) all the other formulas in arithmetical progression may be obtained. When the three quantities which are given, together with the one that is required, are not found in one of the formulas, (A) and (B), it will be necessary to use both.

EXAMPLES.

1. The first term of an increasing arithmetical series is 5, the common difference 2, and the number of terms 24. What is the last term ?

In this example, $a=5$, $d=2$, and $n=24$. By substituting these values in formula (A), we have,

$$l=5+23\times 2=51.$$

2. The first term of an increasing arithmetical series is 5, the last term 51, and the number of terms 24. What is the sum of the series?

In this example, $a=5$, $l=51$, and $n=24$. By substituting these values in formula (B), we have

$$S=\frac{5+51}{2}\times 24=672.$$

3. Having given the sum of an arithmetical series, the first term, and the common difference, it is required to find the number of terms.

In this example, s, a, and d are given to find n. We must therefore eliminate l from formulas A and B. This may be done by substituting in the place of l in formula B, its value as found in formula A, and we then have,

$$S=\frac{2a+(n-1)d}{2}\times n=\frac{2an+dn^2-dn}{2} \qquad (1)$$

By clearing (1) of fractions and transposing, we have

$$dn^2+(2a-d)n=2S \qquad (2)$$

By Rule I., Art.
$$n=\frac{-2a+d\pm\sqrt{8ds+(2a-d)^2}}{2d} \qquad (3)$$

4. Four numbers are in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is 32. What are the numbers?

In this example, we do not apply formulas (A) and (B). For the purpose of rendering the solution more simple, we will let $2y$=the common difference, and $x-3y$=the first term. Then, $x-3y$, $x-y$, $x+y$, and $x+3y$ will represent the numbers. By the conditions of the question, we have,

$$x-3y+x-y+x+y+x+3y=32, \quad (1)$$

and, $$(x-3y)^2+(x-y)^2+(x+y)^2+(x+3y)^2=276. \quad (2)$$

From the (1), $$4x=32 \quad (3)$$

From the (2), $$4x^2+20y^2=276 \quad (4)$$

From the (3), $$x=8, \quad (5)$$

$$\therefore \quad x^2=64 \quad (6)$$

Substitute the value of x^2 in (4), and we have

$$256+20y^2=276 \quad (7)$$

From (7), we find that $y=1$. Hence the series is 5, 7, 9, 11.

5. There are three numbers in arithmetical progression, and the square of the first added to the product of the other two is 16; the square of the second added to the product of the other two is 14. What are the numbers?

Let $x-y$, x, $x+y$, be the numbers.

By the question, we have $$2x^2-xy+y^2=16 \quad (1)$$

And $$2x^2-y^2=14 \quad (2)$$

Subtracting (2) from (1), $$2y^2-xy=2 \quad (3)$$

Put $x=ny$; then $x^2=n^2y^2$. By substituting these values of x and x^2 in (3) and (2), they become

$$2n^2y^2-y^2=14, \quad (4)$$

And $$2y^2-ny^2=2 \quad (5)$$

From (4), $$y^2=\frac{14}{2n^2-1} \quad (6)$$

From (5), $$y^2=\frac{2}{2-n} \quad (7)$$

$$\therefore \quad \frac{2}{2-n}=\frac{14}{2n^2-1} \quad (8)$$

By reducing (8), $4n^2+14n=30$ (9)

Whence, $n=\frac{3}{2}$, or -5 (10)

$$\therefore\ y=\sqrt{\frac{2}{2-n}}=\pm 2$$

Whence, $x=ny=\pm 3$;

$\therefore$ the numbers are 1, 3, 5; or -5, -3, -1.

6. What is the sum of n terms of the series 1, 2, 3, 4, 5 - - - - ?

Ans. $\frac{n(n+1)}{2}$.

7. What is the sum of n terms of the series 1, 3, 5, 7, 9, 11, &c.? *Ans.* n^2.

8. What is the sum of the series 1, 5, 9, 13, 17, &c. to n terms? *Ans.* $n(2n-1)$.

9. The sum of 9 terms of the series $n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2+$ &c.$=501$; required the value of n.

Ans. $n=3$.

10. The first term of an arithmetical progression is n^2-n+1, and the common difference is 2; prove that the sum of n terms is n^3; thence show that $1^3=1^3$, $2^3=3+5$, $3^3=7+9+11$, &c.

11. If S, S', S'' be the sums of three arithmetic series, $1=$the first term of each, and the respective common differences be 1, 2, 3; prove that $S+S''=2S'$.

12. How many terms of the series 1, 3, 5, 7, &c. must be added together to produce the $(2m)$th power of a given quantity r. *Ans.* r^m terms.

13. If S_1, S_2, S_3 - - - - S_n are the sums of n terms of different arithmetic series, having the same first term, and common differences 1, 2, 3 - - - - respectively, show that S_1, S_2, S_3, &c. are in arithmetic progression; and when that first term is 1, prove that

$$S_1+S_2+S_3\cdots+S_n=\frac{n^2(n^2+3)}{4}.$$

14. Having given the first and last terms, and the sum of an arithmetic series; determine the common difference, and apply it to the case where the first term$=1$, the last$=50$, and the sum $=204$.

Ans. $d=\dfrac{(l+a)(l-a)}{2S-(l+a)}=7.$

15. One hundred stones being placed on the ground, in a straight line, at the distance of two yards from each other, how far will a person travel, who shall bring them one by one to a basket placed at two yards from the first stone?

Ans. 11 miles 840 yards.

16. What is the sum of n terms of the series $1+4+7+10+$ &c.?

Ans. $\dfrac{(3n-1)n}{2}.$

17. The product of five numbers in arithmetical progression is 945, and their sum is 25. Required the numbers.

Ans. 1, 3, 5, 7, 9.

18. There are four numbers in arithmetical progression whose continual product is 1680, and common difference is 4. Required the numbers.

Ans. ± 14, ± 10, ± 6, ± 2.

19. A and B, 165 miles distant from each other, set out to meet each other; A travels one mile the first day, two the second, and so on; B travels 20 miles the first day, 18 the second, 16 the third, and so on. How soon will they meet?

Ans. In 10, or 33 days.

20. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Ans. ± 14, ± 10, ± 6, ± 2.

21. A traveller sets out for a certain place, and travels one mile the first day, two the second, and so on. In five days afterwards another sets out, and travels 12 miles a day. How long must he travel to overtake the first?

Ans. 3, or 10 days.

22. There are n arithmetic means between 3 and 17, and the last one is three times as great as the first; find the number of means. *Ans.* 6.

23. In any arithmetic progression of which a is the first term, and $2a$ the common difference; prove that the number of terms which must be taken to make a sum S, is $\sqrt{\frac{s}{a}}$; S being so assumed that $\frac{S}{a}$ is any square number, but no other.

24. Insert 6 arithmetic means between $\frac{1}{2}$ and $\frac{2}{3}$; and find their sum. *Ans.* ?

GEOMETRICAL RATIO AND PROGRESSION.

(137.) Geometrical Ratio is the quotient which is obtained by dividing one quantity by another. Thus, the ratio of 6 to 12 is $12 \div 6 = 2$. The first term of a ratio is called the *antecedent*, the second, the *consequent.*

(138.) If the ratio of two quantities is equal to the ratio of two other quantities, the four quantities taken together constitute a *proportion.* Thus, if $b \div a = d \div c$, a, b, c, and d constitute the four terms of a proportion, which is generally written $a : b :: c : d$; and it is read, a is to b as c is to d. The first and fourth terms are called the extremes, and the second and third, the means.

(139.) The first and third terms of a proportion are called *antecedents*, the second and fourth, consequents.

(140.) Quantities are in proportion by *alternation,* when the two antecedents form one ratio, and the two consequents form the other ratio of a proportion.

(141.) Quantities are in proportion by *inversion*, when the second is to the first, as the fourth is to the third.

(142.) Four quantities are in proportion by *composition*, when the sum of the first antecedent and consequent is to the first antecedent, or consequent, as the sum of the second antecedent and consequent is to the second antecedent or consequent.

(143.) Four quantities are in proportion by *division*, when the difference of the first antecedent and consequent is to the first antecedent, or consequent, as the difference of the second antecedent and consequent is to the second antecedent, or consequent.

THEOREM I.

(144.) *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

If $a:b::c:d$;

Then will $ad=bc$.

For, by Art. 138, $\frac{b}{a}=\frac{d}{c}$.

Divide bd by each member of this equation, and we have,

$$ad=bc.$$

THEOREM II.

(145.) *If the product of two quantities is equal to the product of two other quantities, the two terms of one of the products may be made the extremes, and the two terms in the other product, the means of a proportion.*

If $ad=bc$, then will $a:b::c:d$.

For, divide bd by each member of this equation, and we have,

$\frac{b}{a}=\frac{d}{c}$, that is, $a:b::c:d$.

THEOREM III.

(146.) *If four quantities are in proportion, they will be in proportion by alternation.*

If, $a:b::c:d$, then will $a:c::b:d$

For, since $a:b::c:d$, $ad=bc$. (Theorem I.)

Divide each member of this equation by dc, and we have,

$\frac{a}{c}=\frac{b}{d}$, or $\frac{c}{a}=\frac{d}{b}$, that is, $a:c::b:d$.

THEOREM IV.

(147.) *If four quantities are in proportion, they will be in proportion by inversion.*

If, $a : b :: c : d$, then will $b : a :: d : c$.

For, since $a : b :: c : d$, $ad = bc$ (Theorem I.)

Divide each member of this equation by bd, and we have,

$$\frac{a}{b} = \frac{c}{d}, \text{ that is, } b : a :: d : c.$$

THEOREM V.

(148.) *If four quantities are in proportion they will be in proportion by composition, or division.*

If $a : b :: c : d$, then will $a \pm b : b :: c \pm d : d$.

Since, $a : b :: c : d$, $ad = bc$.

Add and subtract bd from each member of this equation, and we have,

$$ad \pm bd = bc \pm bd,$$

$$\text{Or,} \quad (a \pm b)d = (c \pm d)b.$$

Therefore, by Theorem II., $a \pm b : b :: c \pm d : d$.

We might also show that, $a \pm b : a :: c \pm d : c$.

THEOREM VI.

(149.) *If there be several sets of proportional quantities, the products of the corresponding terms will be proportional.*

$$\left.\begin{array}{ll} \text{If} & a : b :: c : d, \\ & e : f :: h : g, \\ \text{and} & i : j :: k : l. \\ & \&c., \&c. \end{array}\right\} \text{then will } aei : bfj :: chk : dgl.$$

For, from the first proportion, we have, $\frac{b}{a} = \frac{d}{c}$,

and from the second " " $\frac{f}{e} = \frac{g}{h}$,

and from the third " " $\frac{j}{i} = \frac{l}{k}$.

By taking the product of these equations, we have $\frac{bfj}{aei} = \frac{dgl}{chk}$, that is, $aei : bfj :: chk : dgl$.

THEOREM VII.

(150.) *If four quantities are in proportion, their like powers and roots will also be in proportion.*

If $a:b::c:d$, then will $a^n:b^n::c^n:d^n$,

and, $a^{\frac{1}{n}}:b^{\frac{1}{n}}::c^{\frac{1}{n}}:d^{\frac{1}{n}}$

For, since $a:b::c:d$, $\frac{b}{a}=\frac{d}{c}$.

By taking nth power of this equation, we have, $\frac{b^n}{a^n}=\frac{d^n}{c^n}$;

and by extracting the nth root we have, $\frac{b^{\frac{1}{n}}}{a^{\frac{1}{n}}}=\frac{d^{\frac{1}{n}}}{c^{\frac{1}{n}}}$

Whence, $a^n:b^n::c^n:d^n$,

and, $a^{\frac{1}{n}}:b^{\frac{1}{n}}::c^{\frac{1}{n}}:d^{\frac{1}{n}}$.

THEOREM VIII.

(151.) *If there be any number of proportional quantities, the ratio of any antecedent to its consequent will be equal to the ratio of the sum of all the antecedents to the sum of all the consequents.*

If $a:b::c:d::e:f::g:h$, &c.;

then will $a:b::a+c+e+g:b+d+f+h$

For, since $a:b::c:d$, $ad=bc$ (1) (Theorem I.)

Also, since $\left\{\begin{matrix} a:b::e:f \\ a:b::g:h \end{matrix}\right\}$ we have $\left\{\begin{matrix} af=be & (2) \\ ah=bg & (3) \end{matrix}\right.$

By adding together (1), (2), and (3), we have,

$$ad+af+ah=bc+be+bg \qquad (4)$$

By adding ab to each member of (4) we have,

$$ab+ad+af+ah=ab+bc+be+bg, \qquad (5)$$

or, $a(b+d+f+h)=b(a+c+e+g)$.

Therefore, (Theorem II.) $a:b::a+c+e+g:b+d+f+h$.

GEOMETRICAL PROGRESSION.

(152.) A SERIES of quantities, each of which is derived by multiplying the one which immediately precedes it by a constant quantity, is called a *geometrical progression.* When the constant multiplier is greater than unity, the progression is called an *increasing geometrical progression;* when it is less than unity, the progression is called a *decreasing geometrical progression.* Thus, the series 2, 6, 18, 54, in which each term is obtained by multiplying the preceding term by 3, is an increasing geometrical progression; and the series, 48, 24, 12, 6, 3, in which each term is obtained by multiplying the preceding term by $\frac{1}{2}$, is a decreasing geometrical progression. The constant multiplier is called the *common ratio.* If a be the first term, r the common ratio, the series will be

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \&c.$$

PROBLEM I.

(153.) *To find the nth or general term of a geometrical progression.*

By inspecting the series, a, ar, ar^2, ar^3, ar^4, &c., we see that the first term a is a factor of each term in the series, and that the exponent of r in each term is one less than the number which denotes the place of that term in the series. Thus, the exponent of r in the *third* term is 2, in the *fourth* term 3, and so on. Hence, the exponent of r in the nth term is $n-1$, and the nth term is ar^{n-1}. If we denote this term by l, we shall have

$$l=ar^{n-1} \qquad \text{(A)}$$

PROBLEM II.

(154.) *To find a formula for obtaining the sum of all the terms in any geometrical progression.*

If we denote the sum of the series, $a, ar, ar^2, ar^3, \text{-----} ar^{n-1}$ by S, we shall have

$$S=a+ar+ar^2+ar^3 \text{-----} ar^{n-1}. \qquad (1)$$

Multiply equation (1) by r,

$$Sr= \quad ar+ar^2+ar^3 \text{-----} ar^{n-1}+ar^n. \quad (2)$$

Subtract equation (1) from (2), and

$$Sr-s= \quad ar^n-a, \qquad (3)$$

or, $$S(r-1)= \quad ar^n-a; \qquad (4)$$

Whence, $$S= \frac{ar^n-a}{r-1} \qquad (5)$$

By Prob. I., $$l= \quad ar^{n-1}. \qquad (6)$$

Multiply equation (6) by r,

$$lr= \quad ar^n. \qquad (7)$$

Substitute this value of ar^n in the 5th equation, and we have,

$$S=\frac{lr-a}{r-1} \qquad (B)$$

(155.) In a geometrical progression there are five quantities to be considered, namely, *the first term, the last term, the common ratio, the number of terms, and the sum of all the terms.* Any three of these being given, the other two may be found by the aid of the fundamental formulas A and B. It may be observed, however, that when it is required to find r, having given a, n, s, or n, l, s, the solution in each case will involve an equation of the nth degree, which cannot be solved unless we attribute numerical values to the given letters. Four formulas may be obtained for n which involve logarithms.

(156.) When the progression is decreasing, and the number of terms is infinite, the last term, l, must become 0; hence, in this case, formula (B) becomes

$$S=\frac{-a}{r-1}=\frac{a}{1-r} \qquad (D)$$

EXAMPLES AND PROBLEMS IN PROPORTION AND GEOMETRICAL PROGRESSION.

1. Given $\left.\begin{cases} x^3-y^3 : (x-y)^3 :: 7 : 1 & (1) \\ xy=8 & (2) \end{cases}\right\}$ to find the values of x and y.

By Theorem V., we have from (1)

$$x^3-y^3-(x-y)^3 : (x-y)^3 :: 6 : 1. \qquad (3)$$

By expanding $x-y$, and subtracting the result from x^3-y^3, (3) becomes $3x^2y-3xy^2 : (x-y)^3 :: 6 : 1. \qquad (4)$

Divide the two antecedents in (4) by 3, and then divide each member of the first ratio by $x-y$, and it becomes

$$xy : (x-y)^2 :: 2 : 1 \qquad (5)$$

But $xy=8$, $\quad \therefore \; 8 : (x-y)^2 :: 2 : 1$

By Theorem I., $\quad 2(x-y)^2=8$; whence, $x-y=2$, $\qquad (6)$

Or, $\quad x^2-2xy+y^2=4 \qquad (7)$

Multiplying (2) by 4, $\quad 4xy=32 \qquad (8)$

Adding (7) and (8), $x^2+2xy+y^2=36 \qquad (9)$

$\therefore \; x+y=6 \qquad (10)$

From (10) and (6), we readily find that $x=4$, $y=2$.

2. The sum of a geometric series continued to infinity is 3, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

Let $a=$the first term of the series,

And $r=$the common ratio.

Then $a+ar=$the sum of its first two terms; and by formula D, $\frac{a}{1-r}=$the sum of the series. Hence, by the question we have

$$\frac{a}{1-r}=3,$$

And, $\quad a+ar=2\frac{2}{3}. \qquad (2)$

From the (1), $\quad a+3r=3. \qquad (3)$

From the (2), $\quad 3a+3ar=8. \qquad (4)$

Multiply (3) by a, $\quad a^2+3ar=3a. \qquad (5)$

Subtract (4) from (5), $a^2-3a=3a-8$, $\qquad (6)$

Or, $\quad a^2-6a=-8; \qquad (7)$

Whence, $\quad a=4$, or 2. $\qquad (8)$

By substituting these values of a in equation (3), we readily find that $r=-\frac{1}{3}$, or $+\frac{1}{3}$. Hence, we can have the two following series, either of which will answer the conditions of the question:

$$\left\{\begin{array}{ll} 4-\frac{4}{3}+\frac{4}{9}-\frac{4}{27}+\&c. & (1) \\ 2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\&c. & (2) \end{array}\right\}$$

3. What is the sum of $2+6+18+$&c. to 6 terms?

In this example, $a=2$, $r=3$, $n=6$, and by formula (A), $l=ar^{n-1}=2\times3^5=486$. By formula B, $S=\frac{lr-a}{r-1}=\frac{486\times3-2}{3-1}$ $=728$.

4. What is the sum of $1+\frac{1}{3}+\frac{1}{9}+$&c., continued to infinity?

In this example, $a=1$, $r=\frac{1}{3}$, and by formula D,

$$S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}.$$

5. A number (n) of boys arrange themselves in a right line at equal intervals, the nearest being at a given distance (a) from a fixed station, S. A person, P, walks from S to the first boy; and as soon as he begins to return, the remaining boys move *from* S at the same rate as P, and stop when he comes to S. P advances again to the second boy; and whilst he is returning, the remainder move onward, and halt as before. The same is repeated until P has reached the last boy and returned. Now, if under the same circumstances the boys had moved each time towards S, P would have gone only $\frac{1}{m}$th part of his former distance. Show that the distance of the boys from each other is equal to

$$\frac{a(2^n-m-1)}{n(m+1)-(2^n+m-1)}.$$

S A_1 A_2 A_3 A_4 A_5

Let $a=S\,A_1$, the distance from S to the first boy,
And $x=$the distance of the boys from each other.
A_1, A_2, A_3, &c., denote the position of the boys.

Then we readily find that

$2a = 2a + 2x - 2x$ = the whole dist. P travelled 1st time.

$2a + x$ = dist. of A_2 from S when P had returned to S for the 1st time.

$4a + 2x = 4a + 4x - 2x$ = the whole dist. P travelled 2d time.

$4a + 3x$ = dist. of A_3 from S when P had returned to S for the 2d time.

$8a + 6x = 8a + 8x - 2x$ = the whole dist. P travelled 3d time.

$8a + 7x$ = dist. of A_4 from S when P had returned to S for the 3d time.

$16a + 14x = 16a + 16x - 2x$ = the whole dist. P travelled 4th time.

$\vdots \quad \vdots \quad \vdots$ - - - - - - - - - - - - - - - - -

$2^n a + 2^n x - 2x$ = the whole dist. P travelled nth time.

Now, by finding the sum of all the terms in the second members we shall obtain an expression for the entire distance which P travelled, according to the first supposition made in the question. In finding this sum, it may be observed that the sum of the coefficients of a and x in the first and second columns is the same as the sum of the geometrical series

$$2 + 4 + 8 + \&c. \text{ to } n \text{ terms},$$

in which the first term is 2, the ratio 2, and the last term, 2^n. By Problem (B), Art. 154, we find the sum of this series to be

$$S = 2^{n+1} - 2.$$

Hence, $(2a + 4a + 8a + \&c. \text{ to } n \text{ terms}) + (2x + 4x + 8x + \&c. \text{ to } n \text{ terms}) = (2^{n+1} - 2).(a + x)$.

Again, we find that $-2x$ is repeated n times, and therefore the sum of all the terms in the three columns is equal to

$$(2^{n+1} - 2).(a + x) - 2nx.$$

Now if the boys move towards S, we shall find that

$2a$=	the distance travelled by P the 1st time,					
$2x$=	"	"	"	"	2d	"
$2x$=	"	"	"	"	3d	"
⋮	- - - - - - - - - - - - - - - - - - - -					
$2x$=	"	"	"	"	nth	"

Hence, $(n-1)2x+2a$= the whole distance that P would have travelled, if the boys had moved towards S. Therefore, by the condition of the question, we have

$$m[(n-1)2x+2a]=(2^{n+1}-2)(a+x)-2nx. \quad (1)$$

By dividing each member of this equation by 2, we have

$$m[(n-1)x+a]=(2^n-1).(a+x)-nx \quad (2)$$

From equation (2), we easily find that

$$x=\frac{a(2^n-m-1)}{n(m+1)-(2^n+m-1)}. \quad (3)$$

6. Find two numbers such that the difference of their squares is to the square of their difference as 5 to 1; and their sum is to their product as 5 : 6. *Ans.* 3 and 2.

7. It is required to divide the number 14 into two such parts, that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater as 16 : 9.

Ans. The parts are 8 and 6.

8. It is required to divide the number 18 into two such parts, that the squares of those parts may be to each other, as 25 to 16.

Ans. The parts are 10 and 8.

9. The sum of two numbers is 6, and the sum of their squares is to the square of their difference as 5 to 1. What are the numbers? *Ans.* 4 and 2.

10. The sum of two numbers is 12, and their product is to the sum of their squares as 2 to 5. What are the numbers?

Ans. 8 and 4.

11. The first term of a geometrical series is 2, the common ratio, 3; what is the sixth term? *Ans.* 486.

12. The first term of a geometrical series is 4, the common ratio 2, and the last term, 32. What is the sum of the series? *Ans.* 60.

13. The first term of a geometrical series is 3, the common ratio 2, and the number of terms 4; what is the sum of the series? *Ans.* 45.

14. Given 6, the second term of a geometric progression, and 54, the fourth term; find the first term. *Ans.* 2.

15. The fifth term of a geometric progression is 8 times the second, and the third is 12; find the progression.

Ans. 3, 6, 12, 24, &c.

16. If a be the first term of a geometric series, and b the sum of the first three terms; find the common ratio.

$$Ans.\ r=\frac{-a\pm\sqrt{4ab-3a^2}}{2a}.$$

17. If the common ratio of a geometric series be less than $\frac{1}{2}$; prove that any term is greater than the sum of all that follow.

18. The sum of the second and third terms of a geometric series is 24, and the sum of the two next is 216; determine the first term. *Ans.* 2.

19. If four quantities are in geometrical progression, the sum of the two extremes is greater than the sum of the two means. Required the proof.

20. If P be the product, S the sum, and s the sum of the reciprocals of n quantities in geometrical progression; prove that

$$P^2=\left(\frac{S}{s}\right)^n$$

21. Find the sum of the following geometric series:

(*a*) $13+39+117+$&c. to 7 terms. *Ans.* ?

(*b*) $8+20+50+$&c. to 15 terms. *Ans.* ?

(*c*) $1+\frac{1}{4}+\frac{1}{16}+$&c. to n terms. *Ans.* ?

(*d*) $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+$&c. to n terms. *Ans.* $\frac{2}{3}[(\frac{3}{2})^n-1]$

(*e*) $\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+$&c. to n terms. *Ans.* ?

(*f*) $\frac{1}{5}-\frac{2}{15}+\frac{4}{45}-$&c. to n terms. *Ans.* $-\frac{3}{5}[\frac{1}{5}\pm(\frac{2}{3})^n]$

22. Verify the formula in the 20th example, by employing the first three terms of the progression, 2, 4, 8, 16, &c.

23. Find four geometric means between 1 and 32; and two between 1 and 100; and three between $\frac{1}{9}$ and 9; and three between 2 and $\frac{1}{8}$.

24. If S_1, S_2, S_3, - - - - S_n be the sums of n geometric progressions whose first terms are a, $2a$, $3a$, - - - - na; then will

$$S_1+S_2+\text{-}\text{-}+S^n=\frac{n(n+1)}{2}\cdot\frac{r^n-1}{r-1}\cdot a.$$

25. If a and b be the two first terms of a decreasing geometric series continued to infinity; prove that the sum of the series will be equal to $\frac{a^2}{a-b}$.

26. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers. *Ans.* 1, 3, 9, 27.

27. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

Ans. 54, 18, 6, 2.

28. There are three numbers in geometrical progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

Ans. 3, 6, 12.

29. There are three numbers in geometrical progression whose continued product is 64, and the sum of their cubes is 584. Required the numbers. *Ans.* 2, 4, 8.

30. In a geometric series, if the $(p+q)$th term$=m$, and the $(p-q)$th term$=n$, show that the (p)th term$=\sqrt{mn}$, and that the (q)th term$=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

CHAPTER IX.

INDETERMINATE CO-EFFICIENTS, BINOMIAL THEOREM, SUMMATION OF SERIES, EXPONENTIAL THEOREM, AND LOGARITHMS.

INDETERMINATE CO-EFFICIENTS.

(**157.**) THE method of indeterminate co-efficients is a method of expanding an algebraical expression into an infinite series, in which the terms are arranged according to the powers of one of the variable quantities in that expression. The principle employed in this method is included in the following

THEOREM.

If the series $A+A_1x+A_2x^2+$ *&c. be equal to the series* $B+B_1x+B_2x^2+$ *&c., whatever be the value of* x, *then the co-efficients of the like powers of* x *will be equal to each other; that is,* $A=B$, $A_1=B_1$, $A_2=B_2$, *&c.*

For, since the equation

$$A+A_1x+A_2x^2+A_3x^3+\text{\&c.}=B+B_1x+B_2x^2+B_3x^3+\text{\&c.} \qquad (1)$$

is true for all values of x, we can make $x=0$, and then the equation becomes $A=B$.

By subtracting A from one member, and its equal B from the other member of the first equation, we have

$$A_1x+A_2x^2+A_3x^3+\text{\&c.}=B_1x+B_2x^2+B_3x^3+\text{\&c.} \qquad (2)$$

By dividing equation (2) by x it becomes

$$A_1+A_2x+A_3x^2+\text{\&c.}=B_1+B_2x+B_3x^2+\text{\&c.} \qquad (3)$$

By making $x=0$, equation (3) becomes

$$A_1=B_1 \qquad (4)$$

In the same way it may be shown that $A_2=B_2$, $A_3=B_3$, &c.

By transposing in equation (1) we have

$$(A-B)+(A_1-B_1)x+(A_2-B_2)x^2+(A_3-B_3)x^3+\&c.=0 \qquad (5)$$

Make $A-B=D$, $A_1-B_1=D_1$, $A_2-B_2=D_2$, &c., and equation (5) becomes

$$D+D_1x+D_2x^2+D_3x^3+\&c.=0 \qquad (6)$$

Therefore, in any equation of the form of equation (6), each of the co-efficients must equal nothing.

EXAMPLES.

1. Expand the fraction $\frac{1}{1-2x+x^2}$ into an infinite series.

Assume $\frac{1}{1-2x+x^2}=A+A_1x+A_2x^2+A_3x^3+\&c. \qquad (1)$

By clearing this equation of fractions, we have

$$\begin{aligned}1=A&+A_1x+A_2x^2+A_3x^3+\&c.\\ &-2Ax-2A_1x^2-2A_2x^3-\\ &\qquad\quad +Ax^2+A_1x^3+\end{aligned}$$

Hence, by the Theorem, we have

$$\left.\begin{aligned}A&=1\\ A_1-2A&=0\\ A_2-2A_1+A&=0\\ A_3-2A_2+A_1&=0\\ \&c.&\end{aligned}\right\}\text{ Whence }\left\{\begin{aligned}A&=1\\ A_1&=2A=2\\ A_2&=2A_1-A=3\\ A_3&=2A_2-A_1=4\\ \&c.&\end{aligned}\right.$$

Therefore $\frac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+\&c.$*

2. Expand $\sqrt{1+x}$ into an infinite series.

Assume $\sqrt{1+x}=A+A_1x+A_2x^2+A_3x^3+\&c. \qquad (1)$

By squaring equation (1), we have

$$\left.\begin{aligned}1+x=A^2&+AA_1x+AA_2x^2+AA_3x^3+\&c.\\ &+AA_1x+A_1^2x^2+A_1A_2x^3\\ &\qquad\quad +AA_2x^2+A_1A_2x^3\\ &\qquad\qquad\qquad +AA_3x^3\end{aligned}\right\} \qquad (2)$$

* NOTE.—This result may be obtained by actually dividing 1 by $1-2x+x^2$.

By equating the co-efficients of the like powers of x, we have,

$$\left.\begin{array}{r} A^2=1 \\ 2AA_1=1 \\ 2AA_2+A_1^2=0 \\ 2AA_3+2A_1A_2=0 \\ \&c. \end{array}\right\} \text{Whence} \left\{\begin{array}{l} A=1 \\ A_1=\dfrac{1}{2A}=\dfrac{1}{2} \\ A_2=-\dfrac{A_1^2}{2A}=-\dfrac{1}{2.4} \\ A_3=-\dfrac{2A_1A_2}{2A}=\dfrac{1}{2.8} \\ \&c. \end{array}\right.$$

Therefore $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\dfrac{1}{2.4}x^3+\dfrac{1}{2.8}x^4-\&c.$

3. Decompose $\dfrac{3x-5}{x^2-13x+40}$ into two fractions having simple binomial denominators.

The denominator may be resolved into two factors, $x-8$ and $x-5$. Hence we may assume

$$\frac{3x-5}{x^2-13x+40}=\frac{A}{x-8}+\frac{B}{x-5}=\frac{A(x-5)+B(x-8)}{(x-8)(x-5)}=\frac{(A+B)x-5A-8B}{x^2-13x+40}.$$

By omitting the common denominator, we have

$$3x-5=(A+B)x-5A-8B$$

By equating the co-efficients, we have

$$\left.\begin{array}{r} A+B=3, \\ \text{and}\quad 5A+8B=5, \end{array}\right\} \text{whence} \left\{\begin{array}{l} B=-\frac{10}{3} \\ A=\frac{19}{3} \end{array}\right.$$

Therefore we get

$$\frac{3x-5}{x^2-13x+40}=\frac{A}{x-8}+\frac{B}{x-5}=\frac{19}{3}.\frac{1}{x-8}-\frac{10}{3}.\frac{1}{x-5}.$$

4. Expand $\dfrac{1-x}{1+x}$ into an infinite series.

Ans. $1-2x+2x^2-2x^3+2x^4-$ &c.

5. Expand $\dfrac{1-x}{1+x+x^2}$ into an infinite series.

Ans. $1-2x+x^2+x^3-2x^4+x^5+x^6-2x^7+$ &c.

6. Expand $\frac{1+2x}{1-x-x^2}$ into an infinite series.

Ans. $1+3x+4x^2+7x^3+$ &c.

7. Expand $\frac{1+2x}{1-3x}$ into an infinite series.

Ans. $1+5x+15x^2+45x^3+$ &c.

8. Expand $\frac{1-x}{1-3x-2x^2}$ into an infinite series.

Ans. $1+2x+8x^2+28x^3+100x^4+$ &c.

9. Expand $\frac{x}{1+x+x^2}$ into an infinite series.

Ans. $x-x^2+x^4-x^5+x^7-$ &c.

10. Expand $\frac{x}{(1-x)^3}$ into an infinite series.

Ans. $x+3x^2+6x^3+9x^4+$ &c.

11. Resolve $\frac{x+2}{x^3-x}$ into partial fractions.

Ans. $\frac{1}{2(x+1)}+\frac{3}{2(x-1)}-\frac{2}{x}.$

12. Resolve $\frac{2x+3}{x^3+x^2-2x}$ into partial fractions.

Ans. $-\frac{3}{2x}-\frac{1}{6(x+2)}+\frac{5}{3(x-1)}.$

13. Resolve $\frac{1}{x^4-1}$ into partial fractions.

Ans. $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}.$

14. Resolve $\frac{x^2}{(x+1)(x+2)(x+3)}$ into partial fractions.

Ans. $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}.$

15. Resolve $\dfrac{6x^2-22x+18}{x^3-6x^2+11x-6}$ into partial fractions.

$$Ans.\ \frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}.$$

16. Resolve $\dfrac{1+x^2}{x-x^3}$ into partial fractions.

$$Ans.\ \frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}.$$

BINOMIAL THEOREM.

(158.) The *Binomial Theorem* is a Theorem for developing into a series, a binomial having any exponent.

(159.) It was shown in a preceding article that x^n-y^n is divisible by $x-y$ when n is a positive whole number. If we actually divide x^n-y^n by $x-y$ we shall find that

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2\ \text{- - - - - -}\ +xy^{n-2}+y^{n-1}.$$

From the law which governs this quotient, we see that there are n terms in it, and that if $x=y$, each term becomes x^{n-1}. Therefore, by making this supposition, we have

$$\frac{x^n-y^n}{x-y}=nx^{n-1}.$$

Now, whatever be the value of n, we wish to prove that when $x=y$, $(x^n-y^n)\div(x-y)=nx^{n-1}$. There remain then, three other cases to be examined, namely, *when n is a negative whole number, a positive fraction,* and *when n is a negative fraction.*

CASE I.

When n is a negative whole number.

In this case, $(x^n-y^n)\div(x-y)$ becomes

$$\frac{x^{-n}-y^{-n}}{x-y}=\frac{\frac{1}{x^n}-\frac{1}{y^n}}{x-y}=\frac{1}{x^ny^n}\times\frac{y^n-x^n}{x-y}=-x^{-n}y^{-n}\times\frac{x^n-y^n}{x-y}.$$

This becomes when $x=y$, $-x^{-2n}\times nx^{n-1}=-nx^{-n-1}$.

CASE II.

When n is a positive fraction.

Let $n=\frac{r}{s}$, then $(x^n-y^n)\div(x-y)$ becomes

$$\frac{x^{\frac{r}{s}}-y^{\frac{r}{s}}}{x-y}$$

Let $p=x^{\frac{1}{s}}$ and $q=y^{\frac{1}{s}}$ Whence $\left\{\begin{array}{l} p^r=x^{\frac{r}{s}} \\ q^r=y^{\frac{r}{s}} \\ p^s=x \\ q^s=y \end{array}\right\}$ Therefore $\frac{x^{\frac{r}{s}}-y^{\frac{r}{s}}}{x-y}=\frac{p^r-q^r}{p^s-q^s}$.

Divide the numerator and denominator of the last fraction by $p-q$, and it becomes

$$\frac{\frac{p^r-q^r}{p-q}}{\frac{p^s-q^s}{p-q}}$$

Since r and s are positive whole numbers, the last fraction becomes, when $p=q$,

$$\frac{rp^{r-1}}{sp^{s-1}}=\frac{r}{s}p^{r-s}=\frac{r}{s}\left(x^{\frac{1}{s}}\right)^{r-s}=\frac{r}{s}x^{\frac{r}{s}-1}.$$

CASE III.

When n is a negative fraction.

Let $n=-\frac{r}{s}$, then $(x^n-y^n)\div(x-y)$ becomes

$$\frac{x^{-\frac{r}{s}}-y^{-\frac{r}{s}}}{x-y}$$

Let $p=x^{\frac{1}{s}}$ And $q=y^{\frac{1}{s}}$ Whence $\left\{\begin{array}{l} p^{-r}=x^{-\frac{r}{s}} \\ q^{-r}=y^{-\frac{r}{s}} \\ p^s=x \\ q^s=y \end{array}\right\}$ Therefore $\frac{x^{-\frac{r}{s}}-y^{-\frac{r}{s}}}{x-y}=\frac{p^{-r}-q^{-r}}{p^s-q^s}$.

Divide the numerator and denominator of the last fraction by $p-q$, and it becomes

$$\frac{\dfrac{p^{-r}-q^{-r}}{p-q}}{\dfrac{p^{s}-q^{s}}{p-q}}$$

Since r is a negative whole number, and s a positive whole number, the last fraction becomes, when $p=q$,

$$\frac{-rp^{-r-1}}{sp^{s-1}}=\frac{-r}{s}p^{-r-s}=-\frac{r}{s}\left(x^{\frac{1}{s}}\right)^{-r-s}=-\frac{r}{s}x^{-\frac{r}{s}-1}.$$

INVESTIGATION OF THE BINOMIAL THEOREM.

(160.) *It is required to expand into a series $(a+x)^n$, n having any value.*

Since $a+x=a\left(1+\dfrac{x}{a}\right)$, it follows that $(a+x)^n=a^n\left(1+\dfrac{x}{a}\right)^n$.

Therefore, if we expand $\left(1+\dfrac{x}{a}\right)^n$ into a series, and then multiply the result by a^n, we shall have the expansion of $(a+x)^n$.

Let $x_1=\dfrac{x}{a}$, then

$$\left(1+\frac{x}{a}\right)^n=(1+x_1)^n.$$

If we assume $(1+x_1)^n=A+Bx_1+Cx_1^2+Dx_1^3+\&c.$, we shall find that $A=1$. For, since the equation is true for all values of x_1, we may make $x_1=0$, and then the equation becomes $A=1$. Hence, the first term of the expansion of $(1+x_1)^n$ is 1.

Assume, then, $(1+x_1)^n=1+Ax_1+Bx_1^2+Cx_1^3+Dx_1^4+\&c.$ (1)

Putting y for x_1, $(1+y)^n=1+Ay+By^2+Cy^3+Dy^4+\&c.$ (2)

Subtracting (2) from (1), we have

$$(1+x_1)^n-(1+y)^n=A(x_1-y)+B(x_1^2-y^2)+C(x_1^3-y^3)$$
$$+D(x_1^4-y^4)+\&c. \qquad (3)$$

Let $\left.\begin{array}{l} r=1+x_1, \\ \text{and } s=1+y, \end{array}\right\}$ Whence $\left\{\begin{array}{l} r^n=(1+x_1)^n, \\ s^n=(1+y)^n, \end{array}\right\}$ and $x_1-y=r-s$.

By substituting in equation (3) the values of $(1+x_1)^n$ and $(1+y)^n$, it becomes

$$r^n-s^n=A(x_1-y)+B(x_1^2-y^2)+C(x_1^3-y^3)+D(x_1^4-y^4) \\ +\&c. \qquad (4)$$

Divide the first member of equation (4) by $r-s$, and the second, by its equal, x_1-y, and it becomes

$$\frac{r^n-s^n}{r-s}=A+B(x_1+y)+C(x_1^2+x_1y+y^2)+D(x_1^3+x_1^2y+x_1y^2 \\ +y^3)+\&c. \qquad (5)$$

Make $x_1=y$, whence $r=s$, and then equation (5) becomes

$$nr^{n-1}=A+2Bx_1+3Cx_1^2+4Dx_1^3+\&c. \qquad (6)$$

Multiply (6) by r, $nr^n=Ar+2Brx_1+3Crx_1^2+4Drx_1^3+\&c.$ (7)

Or, $$n(1+x_1)^n=A+2Bx_1+3Cx_1^2+4Dx_1^3+\&c. \\ +Ax_1+2Bx_1^2+3Cx_1^3. \qquad (8)^*$$

Multiply eq. (1) by n, $n(1+x_1)^n=n+nAx_1+nBx_1^2+nCx_1^3 +\&c.$ (9)

Since the left hand members of equation (8) and (9) are identical, the right hand members are equal, and by equating the co-efficients of the like powers of x_1, we have

$$\left.\begin{array}{r} A=n \\ 2B+A=nA \\ 3C+2B=nB \\ 4D+3C=nC \\ \&c. \quad \&c. \end{array}\right\} \text{Whence} \left\{\begin{array}{l} A \qquad\qquad\qquad =n, \\ B=\frac{nA-A}{2}=\frac{A(n-1)}{2}=\frac{n(n-1)}{1.2}, \\ C=\frac{nB-2B}{3}=\frac{B(n-2)}{3}=\frac{n(n-1).(n-2)}{1.2.3}, \\ D=\frac{nC-3C}{4}=\frac{C(n-3)}{4}=\frac{n(n-1).(n-2).(n-3)}{1.2.3.4}, \\ \qquad \&c. \qquad\qquad \&c. \end{array}\right.$$

By writing in equation (1), in the place of x_1, its value $\frac{x}{a}$, and

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* NOTE.—Equation (8) is obtained by substituting $1+x_1$, for r in equation (7).

then substituting for A, B, C, D, their values, that equation becomes

$$\left(1+\frac{x}{a}\right)^n=1+n\frac{x}{a}+\frac{n(n-1)}{1.2}\cdot\frac{x^2}{a^2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{x^3}{a^3}+\&c. \quad (10)$$

Multiply equation (10) by a^n, and we have

$$a^n\left(1+\frac{x}{a}\right)^n=(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1.2}a^{n-2}x^2+$$
$$\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^4+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^{n-3}x^4+\&c. \quad (A)$$

By the aid of formula (A) we may expand into a series any binom al. The student cannot fail to notice the law of the exponents, and al-o that of the co-efficients. In applying the formula, it will be necessary to pay strict attention to the *algebraical signs, plus and minus.*

(161.) If in formula (A), we make $a=1$, and $x=1$, we shall have

$$(a+x)^n=(2)^n=1+n+\frac{n(n+1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+$$
$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}+\&c.$$ That is,

The sum of the co-efficients in the expansion of any binomial, whose terms are both positive, is equal to 2 raised to a power denoted by the exponent of the binomial.

Thus, the sum of the co-efficients in the expansion of $(a+x)^4$ is $2^4=16$. This result may be easily verified.

It may also be shown that the sum of the co-efficients in the expansion of a binomial of the form of $(a-x)^n$ is equal to 0.

EXAMPLES.

1. Expand $(a-x)^5$.

Here $n \quad =5,$

$$\frac{n(n-1)}{1.2}=\frac{5\times 4}{2} \quad =10,$$

$$\frac{n(n-1)(n-2}{1.2.3}=\frac{5\times 4\times 3}{2.3} \quad =10,$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}=\frac{5\times4\times3\times2}{1.2.3.4}=5,$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}=\frac{5\times4\times3\times2\times1}{1.2.3.4.5}=1.$$

Whence $(a-x)^5=a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

Since x is negative, all the terms which contain the *odd* powers of x must be negative.

2. Expand $(a+x)^{\frac{1}{3}}$ into a series.

For the sake of clearness we will write out the literal part of the series, according to the law which has been established for the exponents. Thus, we have

$$a^{\frac{1}{3}},\ a^{-\frac{2}{3}}x,\ a^{-\frac{5}{3}}x^2,\ a^{-\frac{8}{3}}x^3,\ a^{-\frac{11}{3}}x^4,\ a^{-\frac{14}{3}}x^5,\ \&c.$$

By the law, the exponent of a in the first term is $\frac{1}{3}$, and in the second term it is 1 less, or $\frac{1}{3}-1=-\frac{2}{3}$. In the third term the exponent of a is 1 less than in the second, or $-\frac{2}{3}-1=-\frac{5}{3}$, and so on. There is no difficulty in regard to the exponents of x. We will now find the co-efficients. In this example $n=\frac{1}{3}$.

Hence n	$=+\frac{1}{3}$,	co-effi. of 2d term.
$\frac{n(n-1)}{1.2}$	$=-\frac{2}{3.6}$,	" 3d "
$\frac{n(n-1)(n-2)}{1.2.3}$	$=+\frac{2.5}{3.6.9}$,	" 4th "
$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$	$=-\frac{2.5.8}{3.6.9.12}$,	" 5th "
$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}$	$=+\frac{2.5.8.11}{3.6.9.12.15}$,	" 6th "
&c.	&c.	&c.

The co-efficient of the third term is obtained by multiplying the co-efficient in the second term by the exponent of a in that term, and dividing the result by the exponent of x in the second term, increased by unity; that is, it is equal to $\frac{1}{3}\times-\frac{2}{3}\div2=-\frac{2}{3.6}$. This co-efficient may be reduced to $-\frac{1}{9}$, but it is better to make

the co-efficients yield to some law, so that after the first two or three are calculated, the others may be readily written out by this law. By writing before each term its co-efficient, we have

$$(a+x)^{\frac{1}{3}}=a^{\frac{1}{3}}+\tfrac{1}{3}a^{-\frac{2}{3}}x-\frac{2}{3.6}a^{-\frac{5}{3}}x^2+\frac{2.5}{3.6.9}a^{-\frac{8}{3}}x^3-\frac{2.5.8}{3.6.9.12}a^{-\frac{11}{3}}x^4$$
$$+\&c.$$

By changing the quantities having negative exponents into the denominator, we have

$$(a+x)^{\frac{1}{3}}=a^{\frac{1}{3}}+\frac{x}{3a^{\frac{2}{3}}}-\frac{2x^2}{3.6a^{\frac{2}{3}}}+\frac{2.5x^3}{3.6.9a^{\frac{8}{3}}}-\frac{2.5.8x^4}{3.6.9.12a^{\frac{11}{3}}}+\&c.$$

By removing the factor $a^{\frac{1}{3}}$, we have

$$(a+x)^{\frac{1}{3}}=a^{\frac{1}{3}}\left(1+\frac{x}{3a}-\frac{2x^2}{3.6a^2}+\frac{2.5x^3}{3.6.9a^3}-\frac{2.5.8x^4}{3.6.9.12a^4}+\&c.\right)$$

If we make $a=8$, and $x=2$, we shall have for the cube root of $a+x$, or 10,

$$10^{\frac{1}{3}}=2\left(1+\frac{2}{3.8}-\frac{2.2^2}{3.6.8^2}+\frac{2.5.2^3}{3.6.9.8^3}-\frac{2.5.8.2^4}{3.6.9.12.8^4}+\&c.\right)$$

By finding the sum of the first four terms of this series, and multiplying the result by 2, we shall obtain for the cube root of 10, 2.153. In order to confirm this result, the pupil may extract the cube root of 10 by the ordinary arithmetical rule.

3. Expand $(a+x)^{\frac{1}{2}}$ into a series.

$$Ans.\ a^{\frac{1}{2}}\left(1+\frac{x}{2a}-\frac{x^2}{2.4a^2}+\frac{1.3x^3}{2.4.6a^3}-\frac{1.3.5x^4}{2.4.6.8a^4}+\&c.\right)$$

4. Expand $\sqrt{a^2-a^2e^2}$ into a series.

$$Ans.\ a\left(1-\tfrac{1}{2}e^2-\frac{e^4}{2.4}-\frac{1.3e^6}{2.4.6}-\frac{1.3.5e^8}{2.4.6.8}-\&c.\right)$$

5. Expand $\frac{1}{(c+x)^2}$, or its equal $(c+x)^{-2}$, into a series.

$$Ans.\ \frac{1}{c^2}\left(1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+\frac{5x^4}{c^4}-\&c.\right)$$

6. Expand $(1+x)^{-\frac{1}{5}}$ into a series.

$$Ans.\ 1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{5.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-\&c.$$

7. Expand $(a^2-x^2)^{\frac{3}{4}}$ into a series.

$$Ans.\ \frac{1}{a^{\frac{1}{2}}}\left(a^2-\frac{3x^2}{2^2}-\frac{3x^4}{2^5a^2}-\frac{5x^6}{2^7a^4}-\frac{5.9x^8}{2^{11}a^6}-\&c.\right)$$

8. Expand $(1+1)^{\frac{1}{2}}$, or $\sqrt{2}$, into a series.

$$Ans.\ 1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}-\&c.$$

9. Expand $(27+1)^{\frac{1}{3}}$, $\sqrt[3]{28}$, into a series.

$$Ans.\ 3\left(1+\frac{1}{3.27}-\frac{2}{3.6.27^2}+\frac{2.5}{3.6.9.27^3}-\frac{2.5.8}{3.6.9.12.27^4}+\&c.\right)$$

10. Expand $(c^2-x^2)^{\frac{3}{4}}$ into a series.

$$Ans.\ c^{\frac{3}{2}}\left(1-\frac{3x^2}{2^2c^2}-\frac{3x^4}{2^5c^4}-\frac{5x^6}{2^7c^6}-\frac{5.9x^8}{2^{11}c^8}-\&c.\right)$$

11. Expand $(a-x)^{-\frac{2}{3}}$ into a series.

$$Ans.\ \frac{1}{a^{\frac{2}{3}}}\left(1+\frac{2x}{3a}+\frac{2.5x^2}{3.6a^2}+\frac{2.5.8x^3}{3.6.9a^3}+\&c.\right)$$

12. Expand $a\left(1+\frac{r}{x}\right)^x$ into a series, and find what the series becomes when $x=\infty$.

$$Ans.\ a\left(1+r+\frac{r^2}{1.2}+\frac{r^3}{1.2.3}+\frac{r^4}{1.2.3.4}+\&c.\right)$$

SUMMATION OF SERIES.

DIFFERENTIAL METHOD.

(162.) THE *differential method* is a method for finding any term, or the sum of any number of terms, of a series in which the successive terms increase according to some law.

Let a, b, c, d, e, &c. be the terms of an increasing series. Now, if we take the first term from the second, the second from the third, and so on, the successive remainders will form a new series, which is called *the first order of differences.* Thus, the first order of differences is

$$b-a,\ c-b,\ d-c,\ e-d,\ \&c. \qquad (1)$$

In the same way we form, from the first order of differences, a series which is *the second order of differences.* Thus, we have for *the second order*,

$$c-2b+a,\ d-2c+b,\ e-2d+c,\ \&c. \qquad (2)$$

In a similar manner we obtain for *the third order of differences*,

$$d-3c+3b-a,\ e-3d+3c-b,\ \&c. \qquad (3)$$

By putting D_1, D_2, D_3, &c. for the first terms of the 1st, 2d, 3d, &c., order of differences, we have

$$\left.\begin{array}{l} D_1=b-a \\ D_2=c-2b+a \\ D_3=d-3c+3b-a \\ D_4=e-4d+6c-4b+a \\ \quad\&c. \end{array}\right\} \text{Whence} \left\{\begin{array}{l} b=a+D \\ c=a+2D_1+D_2 \\ d=a+3D_1+3D_2+D_3 \\ e=a+4D_1+6D_2+4D_3 \\ \quad +D_4,\ \&c. \end{array}\right.$$

Now, let it be observed that the co-efficients of the several terms which compose the value of e, the *fifth* term in the series, a, b, c, d, &c., are the same as the co-efficients of the terms of the expansion of the *fourth* power of a binomial, and that a similar observation may be made in regard to each of the other terms in the series. Therefore, if we represent $(n+1)$th term by P_{n+1}, we shall have

$$P_{n+1}=a+nD_1+\frac{n(n-1)}{1.2}D_2+\frac{n(n-1)(n-2)}{1.2.3}D_3+\&c.$$

By substituting $n-1$ for n, we obtain for the nth term

$$P_n=a+(n-1)D_1+\frac{(n-1).(n-2)}{1.2}D_2+\frac{(n-1).(n-2).(n-3)}{1.2.3}$$
$$+\&c. \qquad (A)$$

(163.) We will now obtain a formula for finding the sum of n terms of an increasing series. For this purpose take the two series

$$a,\ b,\quad c,\quad d,\quad e,\quad \&c., \qquad (1)$$

and

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c. \qquad (2)$$

It is plain that the sum of n terms of the *first* series is equal to the $(n+1)$th term of the *second* series. Hence, we must find the $(n+1)$th term of the *second* series for the sum of n terms of the *first* series. Representing the first terms of the 1st, 2d, 3d, &c., order of differences in the second series by a, D_1, D_2, D_3, &c., we shall have, by what was shown in the preceding article,

$$P_{n+1}=0+na+\frac{n(n-1)}{1.2}D_1+\frac{n(n-1)(n-2)}{1.2.3}D_2+\&c.$$

Now if we represent the sum of n terms of the series a, b, c, d, &c., by S_n, we shall have, since P_{n+1} represents the $(n+1)$th term of the second series,

$$S_n=na+\frac{n(n-1)}{1.2}D_1+\frac{n(n-1)(n-2)}{1.2.3}D_2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$$
$$+\&c. \qquad (B)$$

EXAMPLES.

1. What is the sum of n terms of the series 1, 2, 3, 4, &c.?

$$\begin{array}{ccc} 1, & 2, & 3,\ \&c. \\ & 1 \quad 1 & \\ & 0 & \end{array}$$

Hence, $a=1$, $D_1=1$, $D_2=0$. By substituting these values in formula (B), we have

$$S_n=n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}.$$

This result is the same as that which is obtained by the rules given in arithmetical progression. See example 6, in arithmetical progression.

2. What is the sum of n terms of the series 2, 6, 10, 14, &c.?

$$\begin{array}{ccccc} 2, & & 6, & & 10, \&c. \\ & 4, & & 4 & \\ & & 0 & & \end{array}$$

Hence, $a=2$, $D_1=4$, $D_2=0$. By substituting these values in formula (B), we obtain

$$S_n=2n+\frac{n(n-1)}{2}\times 4=2n^2.$$

3. What is the sum of n terms of the series 1, 2^3, 3^3, 4^3, &c.?

Ans. $\frac{n^2(n+1)^2}{4}$.

4. What is the sum of n terms of the series 1, 3, 5, 7, 9, &c.?

Ans. n^2.

5. What is the sum of the series 1^2, 2^2, 3^2, 4^2, &c. to n terms?

Ans. $\frac{n(2n+1)(n+1}{6}$.

6. What is the sum of n terms of the series $1.2.3+2.3.4+3.4.5+$&c.?

Ans. $\frac{1}{4}n(n+1)(n+2)(n+3)$.

7. What is the sum of 15 terms of the series 1, 4, 8, 13, 19, &c.

Ans. $\frac{1}{6}n(n^2+6n-1)$.

8. What is the sum of n terms of the series 1, 2^4, 3^4, 4^4, 5^4, &c.?

Ans. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

9. What is the sum of n terms of the series 1^2, 3^2, 5^2, 7^2, &c.?

Ans. $\frac{n(2n+1)(2n-1)}{3}$.

10. What is the sum of n terms of the series 1, 3, 6, 10, 15, &c.?

Ans. $\frac{n(n+1)(n+2)}{6}$.

SUMMATION OF INFINITE SERIES.

(164.) An *infinite series* is one which has an infinite number of terms. These terms generally follow some regular law.

(165.) When each term in an infinite series is less than the preceding one, the series is called a *converging series*, and when each is greater than the preceding one, the series is called a *diverging series.*

(166.) The *summation of a series* consists in finding a finite expression equal to the given series. As all series do not follow the same law, it is evident that no one method will apply in summing the different kind of series.

(167.) Since $\frac{q}{n}-\frac{q}{n+p}=\frac{pq}{n(n+p)}$, we have $\frac{q}{n(n+p)}=\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+p}\right)$.

Therefore we infer that the sum of any series of fractions of the form, $\frac{q}{n(n+p)}$, is equal to $\frac{1}{p}$th the difference of the sums of two series of the forms $\frac{q}{n}$ and $\frac{q}{n+p}$. Hence, whenever this difference can be obtained, the sum of the given series may also be obtained.

(168.) In many cases we can find the sum of a series without employing any particular formula.

EXAMPLES.

1. What is the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$&c. to n terms?

In this example, $p=1$, and $n=1, 2, 3, 4$, &c. in succession. Therefore, we have

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}----+\frac{1}{n(n+1)}=\left\{\begin{matrix}\left(1+\frac{1}{2}+\frac{1}{3}-----\frac{1}{n}\right)\\ -\left(\frac{1}{2}+\frac{1}{3}-----\frac{1}{n}+\frac{1}{n+1}\right)\end{matrix}\right\}$$

$$=1-\frac{1}{n+1}=1, \text{ if } n=\infty.$$

2. What is the sum of n terms of the series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6}+$&c.?

Here, $p=2$, and $n=1, 2, 3, 4$, &c., in succession. Therefore, we have

$$Sn=\frac{1}{2}\left\{\begin{matrix}1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}------\pm\frac{1}{n} \\ \frac{1}{3}-\frac{1}{4}------\pm\frac{1}{n}\mp\frac{1}{n+1}\pm\frac{1}{n+2}\end{matrix}\right\}=\frac{1}{4}\pm\frac{1}{2n+2}\mp\frac{1}{2n+4}=\frac{1}{4}$$

when $n=\infty$. The lower sign must be taken when n is *even*, and the upper sign, when n is *odd*.

3. Find the sum of the series $a+2ar+3ar^2+$&c. to n terms.

Let $S=a+2ar+3ar^2----+nar^{n-1}$ (1)

Whence, $S-a=2ar+3ar^2----+nar^{n-1}$ (2)

Multiplying (2) by r, $r(S-a)=2ar^2----+(n-1)ar^{n-1}+nar^n$ (3)

(1)−(3) gives $S(1-r)+ar=a+2ar+ar^2+ar^3----+ar^{n-1}-nar^n$ (4)

Transposing ar, $S(1-r)=(a+ar+ar^2+ar^3+----+ar^{n-1})-nar^n$ (5)

The sum of the geometrical progression within the parenthesis is, by formula (B), Art. 154, $\frac{ar^n-a}{r-1}=\frac{a(1-r^n)}{1-r}$. Hence, equation (5) may be written

$$S(1-r)=\frac{a(1-r^n)}{1-r}-nar^n, \qquad (6)$$

Or, $$S=a\left\{\frac{1-r^n}{(1-r)^2}-\frac{nr^n}{1-r}\right\} \qquad (7)$$

4. What is the sum of $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}$, &c., to infinity ?

Ans. $\frac{3}{4}$.

5. What is the sum of n terms of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+$&c.

Ans. $\frac{n}{3n+3}+\frac{n}{6n+12}+\frac{n}{9n+27}$.

6. What is the sum of the series in the last example to infinity ? *Ans.* $\frac{11}{18}$.

7. What is the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+$&c. to infinity ? *Ans.* $\frac{1}{12}$.

8. What is the sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+$&c., to infinity ? *Ans.* $\frac{1}{12}$.

9. What is the sum of n terms of the series $1+3r+5r^2+7r^3+9r^4+$&c.

Ans. $\frac{(2n-1)r^n}{r-1}-\frac{2r^n-2r}{(r-1)^2}-\frac{1}{r-1}$.

EXPONENTIAL THEOREM.

(169.) THE *Exponential Theorem* is the development of a^x, according to the ascending powers of x.

In order to apply the binomial theorem in the development of a^x, we will let $a=1+c$; whence, $a^x=(1+c)^x$. By the binomial theorem we find that

$$(1+c)^x=1+xc+\frac{x(x-1)}{1.2}c^2+\frac{x(x-1)(x-2)}{1.2.3}c^3+\text{\&c.}$$

By performing the multiplications indicated in this series, we should have a series of monomials involving x, x^2, x^3, &c.; and by inspecting the series we discover that the co-efficient of x is

$$c-\frac{c^2}{2}+\frac{c^3}{3}-\frac{c^4}{4}+\&c.$$

The co-efficients of x^2, x^3, x^4, &c., are also functions* of c. If we represent the several co-efficients of x by A, B, C, &c., we may assume that $a^x=1+Ax+Bx^2+Cx^3+\&c.$, since we have shown that a^x can be developed in this form. Hence, by regarding y and $x-y$ as two other exponents of a, we may have these equations:

$$a^x=1+Ax+Bx^2+Cx^3+\&c. \tag{1}$$

$$a^y=1+Ay+By^2+Cy^3+\&c. \tag{2}$$

And $$a^{x-y}=1+A(x-y)+B(x-y)^2+C(x-y)^3+\&c. \tag{3}$$

Eq. (1)—eq. (2) gives $$a^x-a^y=A(x-y)+B(x^2-y^2)+C(x^3-y^3)+\&c. \tag{4}$$

From (3) we have, $$a^{x-y}-1=A(x-y)+B(x-y)^2+C(x-y)^3+\&c. \tag{5}$$

(5)$\times a^y$ gives $$a^x-a^y=a^y[A(x-y)+B(x-y)^2+C(x-y)^3+\&c.] \tag{6}$$

By equating right-hand members of (4) and (6), we have

$$A(x-y)+B(x^2-y^2)+C(x^3-y^3)+\&c.=a^y[A(x-y)+B(x-y)^2+C(x-y)^3+\&c.] \tag{7}$$

By dividing (7) by $x-y$, and then making $x=y$, it becomes

$$A+2Bx+3Cx^2+4Dx^3+\&c.=a^yA=a^xA. \tag{8}$$

By substituting for a^x its value as found in equation (1), equation (8) becomes

$$A+2Bx+3Cx^2+4Dx^3+\&c.=A+A^2x+ABx^2+ACx^3+\&c. \tag{9}$$

---

* **Note.—A function of a quantity is any algebraical expression involving the quantity.**

By equating the co-efficients of the like powers of x, we have

$$\left.\begin{aligned} A&=A, \\ 2B&=A^2, \\ 3C&=AB, \\ 4D&=AC, \\ &\text{\&c.} \end{aligned}\right\} \text{Whence} \left\{\begin{aligned} A&=A, \\ B&=\frac{A^2}{1.2}, \\ C&=\frac{A^3}{1.2.3}, \\ D&=\frac{A^4}{1.2.3.4}. \\ &\text{\&c.} \end{aligned}\right.$$

Hence equation (1) becomes

$$a^x=1+Ax+\frac{A^2}{1.2}x^2+\frac{A^3}{1.2.3}x^3+\text{\&c.} \tag{A}$$

Which is the *exponential theorem*, where

$$A=c-\frac{c^2}{2}+\frac{c^3}{3}-\frac{c^4}{4}+\text{\&c.}=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\text{\&c.}$$

LOGARITHMS.

(170.) LOGARITHMS are numbers so constructed that many arithmetical calculations may be greatly facilitated by their use.

(171.) *In a system of logarithms every number is regarded as being a power or root of some number, called the base of the system.* The base of a system is arbitrarily assumed. In the common system of logarithms, the base is 10. If we represent the base of any system by a, and its exponent by x, it is easy to see that, by giving all possible values to x, all real numbers may be generated, that is, a^x may be equal to any number by giving to x a proper value. The value which x must have, in order that a^x may be equal to any given number, *is the logarithm of that number.* If we let N represent any number, we shall have

$$a^x=N.$$

If we make $a=10$, and let $x=0, 1, 2, 3, 4$, &c., in succession, we shall have

$10^0=1$	$\therefore$ 0 is the logarithm of	1 in this system.
$10^1=10$	$\therefore$ 1 " "	10 "
$10^2=100$	$\therefore$ 2 " "	100 "
$10^3=1000$	$\therefore$ 3 " "	1000 "
$10^4=10000$	$\therefore$ 4 " "	10000 "
&c. &c.		&c.

Hence it appears, that the logarithm of any number between 1 and 10 must be between 0 and 1; and the logarithm of any number between 10 and 100 must be greater than 1 and less than 2, and so on.

(**172.**) In common logarithmic tables the decimal part of a logarithm is only found. *The integral part of a logarithm which is called the characteristic, is always one less than the number of integral figures in the given number.* This will appear evident by examining the logarithms of 10, 100, 1000, &c. Thus, in looking for the logarithm of 2970, we find in the table opposite the number 472756, which is the decimal part of the logarithm; and the integral part of the logarithm is 3, since the number of integral figures is 4, and affixing 3 to the decimal part we have for the logarithm of 2970, 3.472756.

(**173.**) By letting $x=-1, -2, -3, -4$, &c., we have

$10^{-1}=\frac{1}{10}$	$\therefore$ the logarithm of	$\frac{1}{10}$ is -1,
$10^{-2}=\frac{1}{100}$	$\therefore$ " "	$\frac{1}{100}$ is -2,
$10^{-3}=\frac{1}{1000}$	$\therefore$ " "	$\frac{1}{1000}$ is -3,
$10^{-4}=\frac{1}{10000}$	$\therefore$ " "	$\frac{1}{10000}$ is -4.
&c. &c.		&c. &c.

Hence it appears, that *the characteristic of the logarithm of a decimal fraction is a negative number, which is one greater than the number of ciphers between the decimal point and the first significant figure.* For example, in looking for the logarithm of 0.00462, we find in the tables opposite to 462 the number 664642. Now, since 0.00462 is less than 0.01 and greater than 0.001, its

logarithm must be between -3 and -4, that is, it must be -3 plus a fraction. This fraction we have found to be 0.664642. Therefore the logarithm of 0.00462 is $\bar{3}.664642$. The negative sign placed over the 3 shows that it is negative. The decimal part of a logarithm is always positive.

(174.) Let M and N be any two numbers, and x and y their respective logarithms. Then, by the definition of a logarithm, we have

$$a^x = M. \qquad (1)$$

$$a^y = N. \qquad (2)$$

Eq. (1)$\times$eq. (2) gives $a^{x+y} = M.N,$ (3)

Therefore the logarithm of the product of two numbers is equal to the sum of the logarithms of the two numbers.

(175.) Divide equation (1) by (2), and we have

$$a^{x-y} = \frac{M}{N} \qquad (4)$$

Therefore the logarithm of the quotient of two quantities is equal to the difference of their logarithms.

(176.) Extract the nth root of each member of equation (1) and we have

$$a^{\frac{x}{n}} = M^{\frac{1}{n}} \qquad (5)$$

Therefore the logarithm of the nth *root of a quantity is equal to the* nth *part of its logarithm.*

(177.) Negative numbers can have no logarithms. For, if in equation (1), M is negative, we have $a^x = -M$. Since a is positive, a^x must be positive for every value of x, and therefore M cannot be negative.

PROBLEM.

(178.) *To find the logarithm of any number.*

Let N be any given number whose logarithm is x, in a system whose base is a; then $a^x=N$. It is required to find x in terms of a and N.

$$\left.\begin{array}{ll}\text{Let} & a=1+m \\ \text{And} & N=1+n\end{array}\right\} \text{Whence} \left\{\begin{array}{r}(1+m)^x=1+n \\ \text{and } (1+m)^{xy}=(1+n)^y\end{array}\right. \tag{1}$$

By expanding each member of equation (1) by the binomial theorem, then cancelling the units, and dividing by y, we have

$$xm+\frac{x(xy-1)}{1.2}m^2+\frac{x(xy-1)(xy-2)}{1.2.3}m^3+\&c.=n+\frac{y-1}{1.2}n^2+\frac{(y-1)(y-1)}{1.2.3.}n^3+\&c. \tag{2}$$

Make $y=0$, and the last equation becomes

$$x(m-\tfrac{1}{2}m^2+\tfrac{1}{3}m^3-\&c.)=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\&c. \tag{3}$$

$$\text{Whence,} \quad x=\frac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\&c.}{m-\frac{1}{2}m^2+\frac{1}{3}m^3-\&c.} \tag{4}$$

But $n=N-1$, and $m=a-1$, and by substituting these values of n and m in equation (4), we have

$$x=\log.\ N=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.} \tag{5}$$

This value of logarithm N cannot be used in calculating the logarithms of numbers, since it would require too much numerical labor. We will, therefore, obtain an expression for the logarithm of any number in a *converging series.*

(**179.**) The reciprocal of the denominator of the fraction in equation (5) is called *the modulus of the system.* It is plain that the value of the modulus of any system depends on the base of that system. If we represent the modulus by M, equation (5) will become

$$\text{Log. } N=\log.\ (1+n)=M(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\&c.) \tag{6}$$

If n is negative, then equation (6) becomes

$$\text{Log. } (1-n)=M(-n-\tfrac{1}{2}n^2-\tfrac{1}{3}n^3-\&c.) \tag{7}$$

Eq. (6)−eq. (7) gives log. $(1+n)-$log. $(1-n)=$

$$2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\&c.) \tag{8}$$

Or, $$\log.\frac{1+n}{1-n}=2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\&c.) \tag{9}$$

Let $n=\frac{1}{2P+1}$, whence $\frac{1+n}{1-n}=\frac{P+1}{P}$. Hence, by substitution, equation (9) becomes

$$\text{Log.}\ (P+1)-\log.\ P=2M\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\&c.\right) \tag{10}$$

Or, $$\log.\ (P+1)=\log.\ P+2M\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^3}+\&c.\right) \tag{11}$$

Since the value of M is arbitrary, we may let $M=1$. This is the value which Lord Napier, the inventor of logarithms, assigned to M. Hence, we have

$$\text{Log.}\ (P+1)=\log.\ P+2\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\&c.\right)$$

By making P equal to 1, 2, 3, 4, &c., in succession, we have

$$\text{Log.}\ 2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\&c.\right) =0.6931472.$$

$$\text{Log.}\ 3=\log.\ 2+2\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^3}+\frac{1}{7.5^7}+\&c.\right)=1.0986123.$$

$$\text{Log.}\ 4=2\ \log.\ 2\ \text{- - - - - - - - - - - - - - - - -}\ =1.3862944.$$

$$\text{Log.}\ 5=\log.\ 4+2\left(\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\frac{1}{7.9^7}+\&c.\right)=1.6094379.$$

$$\text{Log.}\ 10=\log.\ 5+\log.\ 2\ \text{- - - - - - - - - - - - - -}\ =2.3025851.$$

In this way we may compute the Naperian logarithms of all numbers.

(**179.**) We will now show how to obtain the common logarithms of numbers. To compute common logarithms, we must find the value of M, the modulus of the system, when the base $a=10$.

Equation (6) in the preceding Article is

$$\log.\ (1+n)=M(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\&c.)$$

By making $M=1$, we have

$$\text{nap. log. } (1+n)=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\&c.$$

where nap. log. $(1+n)$ represents the Naperian logarithm of $(1+n)$. By dividing the first equation by the second, we have

$$M=\frac{\log.\ (1+n)}{\text{nap. log. } (1+n)}$$

Therefore the modulus of any system is equal to the logarithm of any number in that system, divided by the Naperian logarithm of the same number.

Since the logarithm of the base of any system is 1, the last equation will become, by making $1+n=10$,

$$M=\frac{1}{\text{nap. log. } 10}=\frac{1}{2.3025851}=0.43429448.$$

By substituting this value of M in equation (11) of the preceding Article, we have

$$\log.\ (P+1)=\log.\ P+0.86858896\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}\right.$$
$$\left.+\frac{1}{5(2P+1)^5}+\&c.\right) \qquad \text{(A)}$$

By formula (A) we may construct a table of common logarithms. It will only be necessary to compute the logarithms of prime numbers, for every number may be resolved into its prime factors, and then the sum of the logarithms of its prime factors will be equal to the logarithm of the number itself. Art. 174.

By making P equal to 1, 2, 3, 4, &c., in succession, we have

$$\log.\ 2= \qquad 0.86858896\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\&c.\right)=0.3010300$$

$$\log. 3=\log. 2+0.86858896\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\&c.\right)=0.4771213$$

$$\log. 4=2 \log. 2 \text{ - - - - - - - - - - - - - - - - - - } =0.6020600$$

$$\log. 5=\log. 4+0.86858896\left(\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\&c.\right)=0.6989700$$

$$\log. 10=\log. 2+\log. 5 \text{ - - - - - - - - - - - - - - - } =1.0000000$$

&c. &c.

TO FIND THE BASE OF THE NAPERIAN SYSTEM.

(180.) In the Naperian system the base must have such a value as will satisfy the equation,

$$M=\frac{1}{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3+\&c.}=1 \qquad (1)$$

Whence, $a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\&c.=1 \qquad (2)$

By the exponential theorem

$$a^x=1+Ax+\frac{A^2}{1.2}x^2+\frac{A^3}{1.2.3}x^3+\&c. \qquad (3)$$

where A represents the first member of equation (2). Hence, equation (3) becomes

$$a^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\&c. \qquad (4)$$

Make $x=1$, and then the last equation becomes

$$a=1+1+\tfrac{1}{2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\&c.=2.718281828$$

by taking the sum of 12 terms of the series. This value of a is the base of the Naperian system, and it is generally represented by ε, so that we have

$$\varepsilon=2.718281828.$$

APPLICATION OF LOGARITHMS.

In solving some of the following examples, the student will find it necessary to employ a table of logarithms. Any of the tables in common use may be consulted. Instructions for using logarithmic tables will be found in connection with the tables.

EXAMPLES.

1. Multiply 73 by 85.

Here, the logarithm of 73 in the tables is	1.863323
" " 85 " " "	1.929419
Hence, the logarithm of 73×85 is (Art. 174)	3.792742

The number answering to this logarithm is 6205, which is the product required.

2. What is the fifth root of 385?

By Art. 176 the fifth part of the logarithm of any number is equal to the logarithm of the fifth root of that number. The logarithm of 385 is 2.585461. Therefore, the logarithm of $\sqrt[5]{385}$ is $2.585461\div5=0.517092$. The number answering to this logarithm is 3.289 nearly, which is the fifth root of 385.

3. What is the cube root of 551?

Ans. 8.198 nearly.

4. What is the cube root of 267? *Ans.* 6.439.

5. What is the tenth root of 245? *Ans.* 1.733.

6. What is the value of x in the equation $3^x=45$?

7. Calculate the common logarithm of 17.

Ans. 1.2304489.

8. Given the logarithm of 2 to find the logarithm of 25.

Ans. Log. $25=2-2$ log. 2.

9. Given the logarithms of 2 and 3 to find the logarithm of 22.5. *Ans.* Log. $22.5=1+2$ log. $3-2$ log. 2.

10. Find the value of x in the equation $\frac{ab^x+c}{d}=e$.

Ans. $x=\frac{\text{log. }(de-c)-\text{log. }a}{\text{log. }b}$.

11. Show that $\log. \frac{\sqrt{a^2-x^2}}{(a+x)^2}=\frac{1}{2}[\log. (a-x)-3 \log. (a+x)]$.

12. Show that $\log. \sqrt{a^2-x^2}=\frac{1}{2} \log. (a+x)+\frac{1}{2} \log. (a-x)$.

13. Show that $\log. \sqrt[n]{(a^3-x^3)^m}=\frac{m}{n} \log. (a-x)+\frac{m}{n} \log. (a^2+ax+x^2)$.

INTEREST AND ANNUITIES.

(181.) *Interest* is the sum which is paid by the borrower to the lender for the use of money. It is computed at a certain *per cent. per annum;* that is, the borrower pays the lender so many dollars for the use of \$100 for a year, the amount of interest paid for the use of \$100 for one year, is called the *Rate of Interest*, or the *Rate per cent.* Thus, when \$7 is paid for the use of \$100 for one year, the *rate* per cent. is 7. The money lent is called the *principal*, and the interest added to the principal is called the *amount.*

(182.) When interest is regularly received at stated periods, it is called *Simple Interest;* but when it is not so received, and the interest which accrues on the principal during the first of these periods, is added to the principal, and interest is calculated on this amount for the next stated period, and added to it, and so on for each period, the amount which is thus obtained, is called the amount at *Compound Interest* of the principal for the given time. If from this amount, the principal be subtracted, the remainder is the *Compound Interest.*

(183.) The *present worth* of any sum of money which is due at some future period, without interest, is obviously such a sum as will *amount* to the given sum, in the given time, at the given interest.

(184.) Discount is an allowance which is made for the payment of money before it is due. It is obtained by subtracting the *present worth* from the given sum.

(185.) The interest which is obtained by regarding the interest as being due at the end of every *instant*, and then finding the amount at compound interest, is called *Instantaneous Compound Interest.**

(186.) An *Annuity* is a sum of money which is paid regularly once a year, for a certain number of years. Payments which are made semi-annually, quarterly, monthly, &c., are also called annuities.

(187.) The *present worth* of an annuity which is to continue forever, is a sum, the interest of which is equal to the annuity. The *present worth* of an annuity which is to continue for a given number of years, is a sum, which would, if it were placed out at compound interest during the time for which the annuity was to continue, amount to the same sum, which the payments would amount to, if each was placed out at compound interest from the time it became due till the termination of the annuity.

In treating of interest and annuities, we shall employ the following notation:

Let P=the given principal.
r=the interest of \$1 for one year.
n=the number of years.
R=the amount of \$1 for one year.
I=the interest of P dollars for n years.
S=the amount of P dollars for n years at r per cent.
a=an annuity.
A=the amount of an annuity for n years.
p=the present worth of annuity due in n years.
K=the present worth of a given principal, due in n years, at compound interest.

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* NOTE.—For a very interesting article on the subject of Instantaneous Compound Interest, by Prof. Geo. R. Perkins, see *American Journal of Science*, vol. xlvii., No. 1. See also Perkins' Higher Arithmetic, late edition, and Young's Algebra.

SIMPLE INTEREST.

PROBLEM I.

(188.) *Find the interest on P dollars for n years at r per cent.*

Since the interest on 1 dollar for 1 year is r, the interest on P dollars for the same time is P times r, or Pr; and the interest for n years is n times Pr, or nPr. Therefore we have

$$I = nPr, \qquad (1)$$

$$\text{or, } \log. I = \log. n + \log. P + \log. r.$$

Cor. Since the amount is equal to the principal increased by the interest on that principal for the given time, we have

$$S = P + nPr. \qquad (2)$$

PROBLEM II.

(189.) *To find the present worth of any sum of money, which is due in any given time.*

By formula (2) we find that the amount of a sum P due in n years is

$$S = P + nPr.$$

Therefore, P must be the present worth of the sum S due in n years, and we have

$$P = \frac{S}{1+nr}; \qquad (3)$$

that is, the present worth of any sum is equal to the quotient obtained by dividing the given sum by the amount of 1 *dollar for the given time.*

Cor. Therefore, by Art. 184, the *discount* on a sum S due in n years at r per cent. is

$$S - \frac{S}{1+nr} = \frac{Snr}{1+nr}; \qquad (4)$$

that is, the discount on any sum is equal to the quotient obtained by dividing the interest on that sum for the given time, by the amount of 1 *dollar for the given time.*

COMPOUND INTEREST.

PROBLEM III.

(190.) *To find the amount of any sum for a given number of years, and at a given rate per cent., compound interest being allowed.*

By Problem I., the amount of P dollars for one year is $P+Pr=P(1+r)$. Since compound interest is allowed, we must find the amount of $P(1+r)$ dollars for the next year. Now, the amount of 1 dollar for 1 year is $1+r$, and the amount of $P(1+r)$ dollars is, then, $P(1+r)$ times $1+r$, or $P(1+r)^2$. By similar reasoning, we can show, that the amount of P dollars for three years is $P(1+r)^3$, and hence we conclude, that the amount of P dollars for n years is $P(1+n)^n=PR^n$; and, if we denote this amount by S, we have

$$S=PR^n; \qquad (5)$$

that is, the amount of any sum at compound interest for a given time, is equal to the product of the principal, by the amount of 1 *dollar for* 1 *year, raised to a power denoted by the number of years.*

PROBLEM IV.

(191.) *To find the present worth of any sum due in a given number of years, compound interest being allowed at a given rate per cent.*

By Problem III., we find that the amount of P dollars due in n years is

$$S=PR^n.$$

Therefore, P must be the present worth of a sum S due in n years, and we have

$$P=\frac{S}{R^n}. \qquad (6)$$

that is, the present worth of any sum S is equal to the quotient obtained by dividing this sum by the amount of 1 *dollar for the given time.*

ANNUITIES AT COMPOUND INTEREST.

PROBLEM V.

(192.) *To find the amount of an annuity which is to continue for a given number of years, compound interest being allowed at a given rate per cent.*

The annuity which is due at the end of the first year, will be on interest during the $n-1$ remaining years, and therefore, by Problem III., its amount is aR^{n-1}; and the amount of the annuity which is due at the end of the second year is aR^{n-2}, and so on. Hence, the amount of an annuity a, which is to continue for n years, is

$$A=aR^{n-1}+aR^{n-2}+aR^{n-3} - - - - - - +a.$$

The last equation may be written

$$A=a(1+R+R^2 - - - - - - +R^{n-2}+R^{n-1})$$

The quantities within the parenthesis form a geometrical series, the first term of which is 1, the ratio R, and the last term R^{n-1}. The sum of this series is

$$\frac{R^n-1}{R-1}.$$

Hence, the last equation may be written

$$A=\frac{a(R^n-1)}{R-1}. \qquad (7)$$

PROBLEM VI.

(193.) *To find the present worth of an annuity which is to continue for a given number of years, compound interest being allowed.*

If we let K represent the present value of the annuity, then, by Art. 187, the amount of K for the given number of years, at compound interest, must equal the amount of the annuity. Therefore, we shall have, by Probs. III. and V.,

$$KR^n = \frac{a(R^n-1)}{R-1}$$

$$\therefore \quad K = \frac{a(R^n-1)}{R^n(R-1)} \qquad (8)$$

PROBLEM VII.

(**194.**) *To find the present worth of an annuity which is to commence after a given number of years, and continue for a given number of years.*

Let N denote the number of years that will elapse before the annuity commences, and let n represent the number of years for which the annuity is to continue.

It is plain, that if we subtract the present worth of the annuity a for N years, from its present worth for $N+n$ years, we shall have the present value required.

By Prob. VI., the present worth of a for $N+n$ y'rs is $\frac{a(R^{N+n}-1)}{R^{N+n}(R-1)}$

" " " " " N " is $\frac{a(R^N-1)}{R^N(R-1)}$

Therefore, if we denote the present worth required by K, we have

$$K = \frac{a(R^{N+n}-1)}{R^{N+n}(R-1)} - \frac{a(R^N-1)}{R^N(R-1)} \qquad (1)$$

If we multiply both terms of the last fraction by R^n, the two fractions will have a common denominator, and by subtracting their numerators, we have

$$K = \frac{a(R^n-1)}{R^{N+n}(R-1)} \qquad (9)$$

Cor. 1. Equation (9) may be written in this form, by making a slight reduction,

$$K=\frac{a}{R^N(R-1)}\left(1-\frac{1}{R^n}\right)$$

Now, if $n=\infty$, $\frac{1}{R^n}=0$, and the last equation will become

$$K=\frac{a}{R^N(R-1)} \tag{10}$$

Cor. 2. By making $N=0$, equation (9) becomes

$$K=\frac{a(R^n-1)}{R^n(R-1)} \tag{11}$$

Equation (11) agrees with equation (8), and, under this supposition, it is plain that these equations should be identical.

Cor. 3. If, in equation (10), we make $N=0$, it will become

$$K=\frac{a}{R-1}=\frac{a}{r};$$

that is, the present worth of an annuity, which is to commence immediately, and continue forever, is a sum, the yearly interest of which is equal to the annuity. This result agrees with what was said in Art. 187.

PROBLEM VIII.

If the interest on \$1 *for the xth part of a year is* $\frac{r}{x}$, *what is the amount of* \$1 *for* 1 *year, when* $x=\infty$.

Denote the amount by A, then we have, by Problem III.,

$$A=\left(1+\frac{r}{x}\right)^x$$

Expanding the right-hand member of this equation by the binomial theorem, we have

$$A=1+r+\frac{x(x-1)}{1.2}\cdot\frac{r^2}{x^2}+\frac{x(x-1)(x-2)}{1.2.3}\cdot\frac{r^3}{x^3}+\&c.$$

Since x is equal to infinity, $x-1$, $x-2$, &c., cannot differ, essentially, from x, and therefore we may substitute x for $x-1$, $x-2$, &c., and then the equation will readily become*

$$A=1+r+\frac{r^2}{1.2}+\frac{r^3}{1.2.3}+\&c.$$

By the exponential theorem, we know that the right-hand member of this equation is equal to ε^r.

Therefore, we have $A=\varepsilon^r$,

Or, log. $A=r\times$log. $\varepsilon=r\times$log. $2.7182818=r\times 0.43429448$.

If we make $r=0.07$, then we have

$$\text{Log. } A=0.0304009 6;$$

Whence, $A=1.0725$, nearly.

If we wish to ascertain what must be the rate per cent. when the interest is compounded *every instant*, in order that one dollar may amount to \$1.07, we must make $A=\$1.07$, and then the equation above will become

$$\text{Log. } 1.07=r\times 0.43429448,$$

$$\text{Whence,}\quad r=\frac{\text{log. } 1.07}{0.43429448}=\frac{0.0293838}{0.43429448}=0.0676, \text{ nearly.}$$

EXAMPLES.

1. Required the present worth of \$100 due in 18 months hence, at the rate of 7 per cent. per annum.

In formula (3) Problem II., make $S=100$, $r=0.07$, $n=1\frac{1}{2}=1.5$, and we have for the present worth required

$$P=\frac{100}{1+1.5\times 0.07}=\frac{100}{1.105},$$

Or, Log. $P=$log. $100-$log. $1.105=2-0.043362=1.956638$.

The number answering to this logarithm is 90,497. Therefore, $P=\$90.497$, the present worth required.

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* Note.—See Pierce's Differential Calculus, page 176, paragraph 42. We may find the value of A by a different process of reasoning.

2. What is the amount of \$800 for 12 years, at 7 per cent., compound interest?

In formula (5), Problem III., make $R=1.07$, $P=800$, and $n=12$, and we have

$$S=800\times(1.07)^{12},$$

$$\text{Or, Log. } S=\log.\ 800+12\log.\ 1.07=3.255698.$$

The number answering to this logarithm is 1801.76, nearly. Therefore $S=\$1801.76$, the amount required.

3. In what time will \$275 amount to \$1000, at 6 per cent. compound interest?

In formula 5, Problem III., make $R=1.06$, $S=1000$, and $P=275$, and we have

$$1000=275\times(1.06)^n.$$

$$\text{Or, Log. } 1000=\log.\ 275+n\log.\ 1.06.$$

Whence, $n=\dfrac{\log.\ 1000-\log.\ 275}{\log.\ 1.06}=\dfrac{3-2.439333}{0.025306}=22.16$ years, the time required.

4. At what rate per cent., compound interest, will \$100 amount to \$112.48, in 3 years?

In formula (5), Problem III., make $S=112.48$, $P=100$, and $n=3$, and we have

$$112.48=100\times R^3,$$

$$\text{Or, } R^3=1.1248,$$

$$\text{Or, } 3\log. R=\log.\ 1.1248,$$

$$\text{Or, } \log. R=\frac{\log.\ 1.1248}{3}=0.017025,$$

And, $R=1.04$, nearly, and the rate per cent. is 4.

5. What is the present worth of an annuity of \$500, to last for 40 years, at the rate of $2\frac{1}{2}$ per cent. per annum?

In formula (8), Problem VI., make $a=\$500$, $R=1.025$, $n=40$, and we have

$$K=\frac{500\times[(1.025)^{40}-1]}{(1.025)^{40}\times0.025}$$

Now, $\text{Log.}\ (1.025)^{40}=40\ \log.\ 1.025,$

$$=40\times 0.0107239=0.4289560,$$

$$=\log.\ 2.685072,$$

$$\therefore\quad (1.025)^{40}=2.685072.$$

Also, $$\frac{500}{0.025}=20000.$$

$$\therefore\quad K=20000\times\frac{1.685072}{2.685072},$$

$$=20000\times 0.62757,$$

$$=12551.4\ \text{dollars}.$$

6. What is the amount of \$1000 placed out at compound interest of 5 per cent. for 10 years? *Ans.* \$1628.9.

7. What sum must be placed out at compound interest, at 4 per cent., to amount to \$2000 in 15 years? *Ans.* \$1110.5.

8. In how many years will \$200 amount to \$318.80, at 6 per cent. compound interest? *Ans.* 8 years, nearly.

9. At what rate of compound interest must \$518.30 be placed out, to amount to \$600 in 3 years? *Ans.* 5 per cent.

10. In how many years will a given principal double itself, at 4 per cent. compound interest? *Ans.* 17.67 years.

11. In what time will \$100, at 7 per cent. compound interest, amount to the same sum as \$1000 would at 4 per cent. compound interest? *Ans.*

12. Find the amount of \$1200, placed out at compound interest, at 6 per cent. for 10 years, the interest being converted into principal every half year. *Ans.* \$2167.3.

13. What ought to be given for the lease of an estate for 20 years, of the clear annual rental of \$100, in order that the purchaser may make 8 per cent. of his money? *Ans.* \$981.40, nearly.

14. A usurer lent a person \$600, and drew up for the amount a bond of \$800, payable in 3 years, bearing no interest. What did he take per cent., if compound interest be taken into consideration? *Ans.* 10.064 per cent.

15. How long must a capital a remain at interest, the rate per cent. being p, to become as large as a capital a_1, at p_1 per cent., in n years? *Ans.* $\dfrac{\text{Log. } a_1 + n \text{ log. } p_1 - \text{log. } a}{\text{log. } p}$ years.

16. The present value of a freehold estate of \$100 per annum, subject to the payment of a certain sum (Y) at the end of every two years, is \$1000, allowing 5 per cent. compound interest. Find the sum (Y). *Ans.* Y=\$102.50.

17. What will a capital of \$12000 amount to in 10 years, if it bear 6 per cent. compound interest, the interest being paid half yearly? *Ans.* \$21673.333.

18. A capital of \$800 increased, in the space of 6 years, to \$3600. What did the capital gain per cent.? *Ans.* 28.5, nearly.

19. A capital a is put out at p per cent. interest, and at the expiration of each year the interest is added to the principal, and at the same time it is increased or diminished yearly by the sum b. What will this capital amount to in n years hence?

Ans. $Ap^n \pm \dfrac{b(p^n - 1)}{p-1}$; where the upper sign must be used when b is *added*, and the lower sign when b is *subtracted.* $p = 1 + r$

20. What will a capital of \$3740 amount to in 8 years, if at the end of each year \$450 be added to it, reckoning 4 per cent. compound interest? *Ans.* \$9264.833, nearly.

CHAPTER X.

THEORY OF EQUATIONS.

(195.) A *function* of a quantity is any expression involving that quantity. Thus, $ax^2+cx^3+4x^7$, and x^4+5x^3+2x+4 are functions of x. When an expression involves several quantities, it is a function of those quantities. Thus, $ax^2+5y^2+2xy+y$, and $2x^3=x^4-7y^2+14$ are functions of x and y. Such expressions are generally represented by the symbols $F(x)$, $F(x,y)$, which are read function of x, functions of x and y.

PROPOSITION I.

(196.) *If any function of the form* $x^n+Ax^{n-1}+Bx^{n-2}$ - - - - - - $+P$ *be divided by* $x-a$, *the remainder will be the same function of* a *that the given polynomial is of* x.

Let $F(x)=x^n+ax^{n-1}+bx^{n-2}$ - - - - - - $+p$. Now divide $F(x)$ by $x-a$, and represent the quotient, whatever it may be, by Q; and let R represent the remainder which does not involve x. Since the dividend is equal to the divisor multiplied by the quotient plus the remainder, we have

$$F(x)=Q(x-a)+R$$

This equation is true for all values of x; hence, we may let $x=a$, and the equation then becomes

$$F(a)=0+R$$

But R is independent of x, and therefore the remainder is the same function of a, that the given polynomial is of x.

Cor. It is obvious that whenever we wish to divide a polynomial of the form of $x^n+Ax^{n-1}+Bx^{n-2}$ - - - - - - $+B$ by $x-a$,

we can find the remainder without performing the labor of division. Thus, if we divide the polynomial x^3-6x+7 by $x-2$, we have for the remainder, $R=2^2-6\times2+7=-1$.

PROPOSITION II.

(197.) *If a is a root of the equation $x^n+Bx^{n-1}+Cx^{n-2}$ - - - - $+P=0$, then the equation is divisible by $x-a$.*

If the given equation be divided by $x-a$, and the division is carried on till a remainder is obtained which does not involve x, this remainder must be the same function of a, that the given polynomial is of x. (Prop. I.) Hence, the remainder is $a^n+Ba^{n-1}+Ca^{n-2}$ - - - - - $+P$. But, since a is a root of the given equation, it must verify that equation, and we have $a^n+Ba^{n-1}+Cx^{n-2}$ - - - - - $P=0$; hence, the remainder is equal to zero, and therefore the given equation is exactly divisible by $x-a$.

Cor. If the first member of any equation of the form $x^n+Bx^{n-1}+Cx^{n-2}$ - - - - $P=0$ is divisible by $x-a$, then a is a root of that equation. For, since the given equation is exactly divisible by $x-a$, the remainder, which is the same function of a that the given equation is of x, must be equal to zero; that is, $a^n+Bx^{n-1}+Cx^{n-2}$ - - - - $+P=0$. Hence, it appears that if a be substituted for x in the given equation, the equation is verified, and therefore a is a root of that equation.

PROPOSITION III.

(198.) *Any equation which contains only one unknown quantity has as many roots as there are units in the highest power of the unknown quantity.**

Let $F(x)$ represent any equation of the nth degree, and let a represent a root of this equation. Then, by Proposition II., $F(x)$ is exactly divisible by $x-a$, and if we represent the quotient, which is a function of x, by $F_2(x)$, we shall have, by a principle in division,

$$F(x)=(x-a_1)\times F_2(x)=0 \qquad (1)$$

*** Note.—In this demonstration it is assumed that every equation has at least one root.**

Now, the equation $(x-a_1)\times F_2(x)=0$ may be satisfied by making either of its factors equal to zero. Hence, we may have $F_2(x)=0$. If we represent a root of this equation by a_2, we shall have, by reasoning as before, $F_2(x)=(x-a_2)\times F_3(x)$. Substitute this value of $F_2(x)$ in equation (1), and it becomes

$$F(x)=(x-a_1).(x-a_2)\times F_3(x) \qquad (2)$$

It is plain that we may continue this process of reasoning till we have resolved $F(x)$ into n factors, and no more; hence, the equation, $F(x)=0$, may assume this form

$$F(x)=(x-a_1).(x-a_2).(x-a_3) \text{ - - - - - } (x-a_n)=0 \qquad (3)$$

Whence, it appears that there are as many roots as factors, that is, n roots; for equation (3) may be satisfied by making any one of the n quantities, a_1, a_2, a_3, a_4, - - - - - a_n, equal to x. Therefore the proposition is established.

PROPOSITION IV.

(199.) *If x represents the unknown quantity in any equation whose roots are a_1, a_2, a_3, - - - - - - a_n, then the equation is equal to the continued product of $x-a_1$, $x-a_2$, $x-a_3$, - - - - - - - - $x-a_n$.*

For, if $F(x)$ represents the required equation, we have, by the last Proposition,

$$F(x)=(x-a_1).(x-a_2).(x-a_3) \text{ - - - - - } (x-a)=0$$

Hence the proposition is true.

If we perform the multiplications indicated in the last equation, we shall have, by making n equal 1, 2, 3, 4, &c., the following equations:

When $n=2$. $F(x)=x^2 \left.\begin{matrix}-a_1\\-a_2\end{matrix}\right\} x+a_1a_2=0$

When $n=3$. $F(x)=x^3 \left.\begin{matrix}-a_1\\-a_2\\-a_3\end{matrix}\right\} x^2 \left.\begin{matrix}+a_1a_2\\+a_1a_3\\+a_2a_3\end{matrix}\right\} x+a_1a_2a_3=0$

When $n=4$. $$F(x)=x^4\left.\begin{matrix}-a_1\\-a_2\\-a_3\\-a_4\end{matrix}\right\}x^3\left.\begin{matrix}+a_1a_2\\+a_1a_3\\+a_2a_3\\+a_1a_4\\+a_2a_4\\+a_3a_4\end{matrix}\right\}x^2\left.\begin{matrix}-a_1a_2a_3\\-a_1a_2a_4\\-a_1a_3a_4\\-a_2a_3a_4\end{matrix}\right\}x+a_1a_2a_3a_4=0, \text{ \&c.}$$

By examining these equations, we discover the following properties:

1. The co-efficient of the *second* term in either of the equations is the sum of all the roots with their signs changed.

2. The co-efficient of the *third* term in either of the equations is the sum of all the *different* products that can be obtained from the roots, taken two and two, with their signs changed.

3. The co-efficient of the *fourth* term in either of the equations is the sum of all the *different* products that can be obtained from the roots, taken three and three, with their signs changed; and, by deduction, we conclude that the co-efficient of the nth term is the sum of all the *different* products that can be obtained from the roots, taken $n-1$ and $n-1$, with their signs changed.

4. The *last*, or *absolute* term, is the product of all the roots with their signs changed.

Cor. 1. If the signs of the terms of an equation are alternately positive and negative, the roots are all positive, and if the signs are all positive, then the roots are all negative.

Cor. 2. If the co-efficient of the second term of any equation is 0, that is, if the second term is wanting, the sum of the positive roots is equal to the sum of the negative roots.

Cor. 3. Every root of the equation is a divisor of the last or absolute term.

PROPOSITION V.

(200.) *No equation has an odd number of surd, or an odd number of imaginary roots.*

For, let $x^n+Ax^{n-1}+Bx^{n-2}$ - - - - - - $P=0$ be an equation, one root of which is of the form $a+b\sqrt{-1}$; then will $a-b\sqrt{-1}$ be a root of this equation.

Since $a+b\sqrt{-1}$ is a root of the equation, it must, when substituted for x, satisfy the equation, and we shall have

$$(a+b\sqrt{-1})^n+A(a+b\sqrt{-1})^{n-1}+B(a+b\sqrt{-1})^{n-2} \text{ - - - - - } P=0 \qquad (1)$$

Now, as all *even* powers of an imaginary quantity give real quantities, and all *odd* powers imaginary quantities, it follows that if we expand the terms in the last equation, the resulting equation will contain some terms which are real, and some that are imaginary. If we represent the sum of the real quantities by P, and the sum of the imaginary quantities by $Q\sqrt{-1}$, we shall have

$$P+Q\sqrt{-1}=0 \qquad (2)$$

It is obvious that this last equation can only be satisfied by making P and Q equal to 0 at the same time.

If we now substitute for x in the given equation $a-b\sqrt{-1}$, and then expand the different terms as before, an expression will be obtained which will only differ from the former expanded result in the signs of the odd powers of $b\sqrt{-1}$. Hence, if we represent the former result by $P+Q\sqrt{-1}$, we may represent the latter by $P-Q\sqrt{-1}$. But it has been shown that $P=0$, and $Q=0$. Therefore $P-Q\sqrt{-1}=0$. Hence the substitute of $a-b\sqrt{-1}$ for x verifies the equation, and therefore it is a root of that equation.

In a similar manner it may be shown that if $a+\sqrt{b}$ is a root of an equation, then $a-\sqrt{b}$ is also a root of that equation.

COR. 1. All the roots of an equation of an even degree may be imaginary, but if they are not all imaginary, there must be at least two real roots.

COR. 2. The product of two imaginary roots, $a+b\sqrt{-1}$, $a-b\sqrt{-1}$, is a^2+b^2, a positive quantity; and, therefore, when all the roots are imaginary, the absolute term must be positive.

COR. 3. An equation of an odd degree must have at least one real root, and the sign of this root is contrary to that of the absolute term.

COR. 4. An equation of an even degree, whose last or absolute term is negative, must have at least two real roots.

PROPOSITION VI.

(201.) *If the signs of the alternate terms in any equation be changed, the corresponding roots of the resulting equation will have contrary signs.*

Let $x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}$ - - - - - $+P=0$, be any equation. If we change the signs of the alternate terms, we shall have

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3} \text{ - - - - } \pm P=0; \qquad (1)$$

or, $$-x^n+Ax^{n-1}-Bx^{n-2}+Cx^{n-3} \text{ - - - - - } \mp P=0 \qquad (2)$$

Equations (1) and (2) are identical, for by transposition in equation (1), we have

$$x^n+Bx^{n-2} \text{ - - - - } \pm P=Ax^{n-1}+Cx^{n-1}+\&c. \qquad (3)$$

By transposition in equation (2), we have

$$x^n+Bx^{n-2} \text{ - - - - } \pm P=Ax^{n-1}+Cx^{n-1}+\&c. \qquad (4)$$

Now, as equations (3) and (4) are identical, equations (1) and (2) from which they were derived, must also be identical. Now let a be a root of the given equation, and let a be substituted for x in the given equation, and $-a$ for x in equation (1), and the results in each case will be the same. But since a is a root of the given equation, it must verify it, and hence $-a$ must verify equation (1), and therefore equation (2), equations (1) and (2) being identical. Therefore, $-a$ is a root of equations (1) and (2). Hence the proposition is established.

TRANSFORMATION OF EQUATIONS.

PROBLEM I.

(202.) *To transform any equation into another, so as to remove the second term.*

Let $x^n+Ax^{n-1}+Bx^{n-2}$ - - - - $+P=0$ represent any equation Let $x=r+s$, and then substitute this value of x in the given equation, and we have

$$(r+s)^n+A(r+s)^{n-1}+B(r+s)^{n-2} \text{ - - - - - - - } +P=0$$

By expanding the terms in this equation by the Binomial Theorem, we have

$$\left.\begin{array}{r} r^n+ns \\ +A \end{array}\right\} r^{n-1} \left.\begin{array}{r} +\dfrac{n(n-1)}{1.2}s^2 \\ +(n-1)As \\ +B \end{array}\right\} r^{n-2} \text{ - - - - } \left.\begin{array}{l} +\ s^n \\ +As^{n-1} \\ +Bs^{n-2} \\ \vdots \\ +P \end{array}\right\} =0$$

Now, if we regard s as being an indeterminate quantity, we can attribute to it such a value as will make the co-efficient of r^{n-1} in the last equation equal to 0. Hence, we may have

$$ns+A=0;$$

$$\therefore \quad s=-\frac{A}{n}.$$

But $$x=s+r;$$

$$\therefore \quad x=r-\frac{A}{n}.$$

Hence, the second term of any equation may be made to disappear, by substituting for the unknown quantity another unknown quantity, connected with the co-efficient of the second term, with a contrary sign, and divided by the exponent of the highest power of the unknown quantity in the given equation.

(203.) If we have the quadratic equation $x^2+2ax=b$, we can find the values of x by the aid of the preceding article. Thus, let $x=y-a$, and then substitute this value of x in the given equation, and we have

$$(y-a)^2+2a(y-a)=b,$$

or $$y^2-2ay+a^2+2ay-2a^2=b,$$

or $$y^2-a^2=b,$$

Whence, $$y=\pm\sqrt{a^2+b};$$

$$\therefore \quad x=-a\pm\sqrt{a^2+b}.$$

PROBLEM II.

(204.) *To transform any equation into another whose roots shall be equal to the roots of the given equation increased or diminished by a given quantity.*

Let $x^n+Ax^{n-1}+Bx^{n-2}$ - - - - - - - - $+P=0=X$ be any equation. Substitute for x in this equation $s+r$, and it becomes

$$(s+r)^n+A(s+r)^{n-1}+B(s+r)^{n-2} \text{ - - - - - } +P=0=X.$$

By expanding the terms in this equation by the Binomial Theorem, and arranging them according to the ascending powers of r, we have

$$\left.\begin{array}{ll} s^n & +ns^{n-1} \\ +As^{n-1} & +(n-1)As^{n-2} \\ +Bs^{n-2} & +(n-2)Bs^{n-3} \\ \vdots \quad + & \vdots \\ +P & \end{array}\right\} r \left.\begin{array}{l} +n.\frac{n-1}{1.2}s^{n-2} \\ +(n-1)\frac{n-2}{1.2}As^{n-3} \\ +(n-2)\frac{n-3}{1.2}Bs^{n-4} \\ \\ + \text{ - - - - - - - - - -} \end{array}\right\} r^2 \text{ - - - - - } +r^n = 0 \qquad (A)$$

If, in this equation, we represent the sum of the terms which do not involve r, or what is the same thing, the co-efficient of r^0, by X_1, the co-efficient of r by X_2, the co-efficient of r^2 by $\frac{X_3}{2}$, the co-efficient of r^3 by $\frac{X_4}{2.3}$, and so on, we shall have the following equations:

$$\left.\begin{array}{ll} X_1=s^n+As^{n-1}+Bs^{n-2}\text{ - - - - - - - - - - - - - }+P, & (1) \\ X_2=ns^{n-1}+(n-1)As^{n-2}+(n-2)Bs^{n-3} \qquad +0, & (2) \\ X_3=n(n-1)s^{n-2}+(n-1)(n-2)As^{n-3}+(n-2)(n-3) & \\ \qquad Bs^{n-3}\text{ - - - - }N,\ \&c. & (3) \end{array}\right\} (B)$$

By examining these equations, we find that the first may be derived from the given equation by simply changing x into s; the second may be derived from the first by multiplying each term by the exponent of s in that term, and then diminishing the exponent of s by a unit in each term. The third may be derived from the second in the same manner that the second was derived from the first, and so on. Equations (1), (2), and (3), are called derived polynomials.

If we substitute for the co-efficients of r in equation (A), the quantities which represent them, we shall have

$$X_1+X_2r+\frac{X_3}{2}r^2+\frac{X_4}{2.3}r^3\text{ - - - - - - }+r^n=0. \qquad (C)$$

To show the application of equations (B) and equation (C), let it be required to find an equation whose roots shall be less by 3 than the roots of the equation

$$x^4-3x^3-15x^2+49x-12=0.$$

Let $x=r+3$. It is evident that if we substitute this value of x in the given equation, the roots of s in the resulting equation will be less by 3 than the roots of x in the given equation. The transformed equation will be of the form

$$X_1+X_2r+\frac{X_3}{2}r^2+\frac{X_4}{2.3}r^3+\frac{X_5}{2.3.4}r^4=0 \qquad (m)$$

In this example, $s=3$, and by the aid of equations (B), we find that

$$\begin{array}{ll} X_1=(3)^4-3(3)^3-15(3)^2+49(3)-12=0 & \\ X_2=4(3)^3-9(3)^2-30(3)+49 & =-14 \\ \dfrac{X_3}{2}=6(3)^2-9(3)-15 & =+12 \\ \dfrac{X_4}{2.3}=4(3)-3 & =+9 \\ \dfrac{X_5}{2.3.4}=1 & =+1 \end{array}$$

By substituting these values of the co-efficients in equation (m), it becomes

$$r^4+9r^3+12r^2-14r=0$$

(**205.**) We may also transform an equation into another, whose roots shall be less or greater than the roots of the given equation by a given quantity, by division.

Let $x^n+Ax^{n-1}+Bx^{n-2} \text{ - - - } +Ox+P=0$ (1) be any equation. If we make $x=s+r$, and then substitute this value of x in the proposed equation, the transformed equation will obviously be of the form

$$s^n+A_1s^{n-1}+B_1s^{n-2} \text{ - - - - - } +O_1s+P_1=0 \qquad (2)$$

If, in this equation, we substitute for s its value, $x-r$, it will become

$$(x-r)^n+A_1(x-r)^{n-1}+B_1(x-r)^{n-2} \text{ - - - - } +O_1(x-r)+P_1=0 \quad (3)$$

Equation (3), when developed, must be identical with equation (1). For, equation (2) was obtained by substituting $s+r$ for x in equation (1), and then by substituting $(x-r)$ for s in equation (2) we must needs have returned to the original equation. Hence, we have,

$$(x-r)^n+A_1(x-r)^{n-1}+B_1(x-r)^{n-2} \text{ - - - } +O_1(x-r)+P_1=x^n+ \\ Ax^{n-1}+Bx^{n-2} \text{ - - } Ox+P$$

If we divide the first member of this equation by $x-r$, the remainder will obviously be P_1, and the quotient will be

$$(x-r)^{n-1}+A_1(x-r)^{n-2}+B_1(x-r)^{n-3} \text{ - - - } N_1(x-r)+O_1$$

If we divide the second member by $x-r$, the quotient and remainder must be the same, since the second member is identical with the first. Therefore, if the first member of equation (1) be divided by $x-r$, the remainder is the absolute term in equation (2), which is the required transformed equation. Again, if we divide the quotient already obtained by $x-r$, the remainder will be O_1, the co-efficient of s in the transformed equation. By proceeding in this way, it is plain that we may determine all of the co-efficients of s in the transformed equation.

Hence, we have the following rule for transforming an equation into another, whose roots are equal to the roots of the given equation, increased or diminished by a given quantity.

RULE.

Transpose all the terms of the given equation into the first member, and then divide the given equation by the unknown quantity increased or diminished by a given quantity, according as the roots of the proposed equation are to be increased or diminished, and then divide the quotient last obtained by the same divisor, and so on till a quotient is obtained which is independent of the unknown quantity. Continue the division in each case till the remainder is independent of the unknown quantity.

The co-efficient of the highest power of the unknown quantity in the transformed equation will be the same as the co-efficient of the highest power of the unknown quantity in the proposed equation, and the co-efficients of the following terms in the transformed equation will be the several remainders, taken in a reverse order.

Wherever there is an absent term in the equation, supply its place with a cipher.

We will now apply the above rule in transforming the equation $x^3-7x+7=0$ into another whose roots are less by 1 than those of the given equation. We shall designate the first remainder by R_1, the second by R_2, and so on.

Operation.

$$x-1)x^3-7x+7(x^2+x-6=\text{1st quotient.}$$
$$\underline{x^3-x^2}$$
$$x^2-7x$$
$$\underline{x^2-x}$$
$$-6x+7$$
$$\underline{-6x+6}$$
$$+1=R_1$$

$$
\begin{array}{l}
x-1)x^2+x-6(x+2=\text{2d quotient.} \\
\underline{x^2-x} \\
\quad 2x-6 \\
\quad \underline{2x-2} \\
\qquad -4=R_2
\end{array}
$$

$$
\begin{array}{l}
x-1)x+2(1=\text{3d quotient.} \\
\underline{x-1} \\
\quad +3=R_3
\end{array}
$$

Hence, the required transformed equation is $x^3+3x^2-4x+1=0$.

This method of transforming equations will be of no practical value, on account of its length, unless we have some means of diminishing the labor. To this end we shall here introduce

HORNER'S SYNTHETIC METHOD OF DIVISION.

(206.) We shall first show how division may be performed by means of *detached co-efficients*, when the divisor and dividend are homogeneous, and contain only two letters.

Let it be required to divide $a^3+3a^2x+3ax^2+x^3$ by $a+x$. By examining the divisor and dividend, it will appear evident that the first term in the quotient is a^2, and that the exponent of a is one less in any term than in the term which immediately precedes it; and that the exponent of x in the second term of the quotient is 1, and that the exponent of x is greater by 1 in any term than in the term which immediately precedes it. Hence, the literal parts of the quotient are a^2, ax, x^2. The literal parts of the successive terms in the quotient follow the same law of increase and decrease as those in the dividend. Now, the co-efficients of the terms in the quotient may be determined by writing down the co-efficients of the divisor and dividend, and then proceeding to divide as though the literal parts were attached to them.

Operation.

$$\begin{array}{l}
1+1)1+3+3+1(1+2+1, \text{ co-efficients.} \\
\quad\ \ \underline{1+1} \\
\qquad\ \ 2+3 \\
\qquad\ \ \underline{2+2} \\
\qquad\qquad 1+1 \\
\qquad\qquad \underline{1+1} \\
\qquad\qquad\quad 0
\end{array}$$

Therefore $(a^3+3a^2x+3ax^2+a^3)\div(a+x)=a^2+2ax+x^2$.

As another example, we will divide $x^4-3ax^3-8a^2x^2+18a^3x-8a^4$ by $x^2+2ax-2a^2$.

Operation.

$$\begin{array}{l}
1+2-2)1-3-8+18-8(1-5+4 \\
\qquad\quad \underline{1+2-2} \\
\qquad\quad -5-\ \ 6+18 \\
\qquad\quad \underline{-5-10+10} \\
\qquad\qquad\qquad 4+\ \ 8-8 \\
\qquad\qquad\qquad 4+\ \ 8-8
\end{array}$$

In the following example, there are no terms which contain a^3, a^2, a, or each of the co-efficients of these terms is 0.

Divide $6a^4-96$ by $3a-6$.

Operation.

$$\begin{array}{l}
3-6)6+0+0+0-96(2+4+8+16 \\
\qquad\ \ \underline{6-12} \\
\qquad\qquad 12 \\
\qquad\qquad \underline{12-24} \\
\qquad\qquad\quad 24 \\
\qquad\qquad\quad \underline{24-48} \\
\qquad\qquad\qquad 48-96 \\
\qquad\qquad\qquad 48-96
\end{array}$$

Since $a^4\div a=a^3$, the literal parts of the quotient are a^3, a^2, a; therefore, the quotient is $2a^3+4a^2+8a+16$.

(207.) By the common rule for division, each term in the divisor is multiplied by the first term in the quotient, and the several terms of the product are *subtracted* from the dividend. Now, if we change the signs of the terms in the divisor, and then multiply each term of the divisor by the first term of the quotient, the terms of this product must be *added* to the dividend; and since we can do the same for each term in the quotient, the several terms of the products of the terms in the divisor by the successive terms in the quotient would all become *additive.*

By this process, the first remainder is the same as would be obtained by the ordinary rule, but since the sign of the first term in the divisor has been changed, it is plain that if we divide the first term of the remainder by the first term of the divisor we shall obtain the next term in the quotient with its sign *changed.* Now, to avoid the liability of error, incident to this change of sign in the quotient, we may let the first term of the divisor remain unchanged, and then omit altogether the products of the first term of the divisor by the successive terms in the quotient, since by the usual method of division, the first term of the dividend, and the first terms of the several remainders are cancelled by these products.

Let it be required to divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. By detaching the co-efficients, we shall have the following

Operation.

$$
\begin{array}{l}
1+2-1)1-5+10-10+5-1(1-3+3-1 \\
\qquad\qquad\quad {*}+2-\ 1 \\
\qquad\qquad\quad \overline{(-3+\ 9-10)}=\text{1st remainder.} \\
\qquad\qquad\qquad\quad {*}-\ 6+\ 3 \\
\qquad\qquad\qquad\quad \overline{(+\ 3-\ 7+5)}=\text{2d remainder.} \\
\qquad\qquad\qquad\qquad\quad {*}+\ 6-3 \\
\qquad\qquad\qquad\qquad\quad \overline{(-\ 1+2-1)}=\text{3d remainder.} \\
\qquad\qquad\qquad\qquad\qquad\ {*}-2+1
\end{array}
$$

This operation may be abbreviated by omitting all the additions except those which must be performed for the purpose of obtaining the first terms in the remainders, which are the successive terms in the quotient. For example, instead of adding -1 to 10, and then adding to their sum -6, we may omit the first addition, and find the sum of $+10-1-6$, at once. For the sake of convenience in performing the additions, multiply the changed terms in the divisor by each term in the quotient, as it is found, and then arrange the terms of each series of products in a diagonal line passing downwards. The first terms of the several series of products are arranged immediately under the terms of the dividend. The last operation may be modified as follows:

$$\begin{array}{r}
1+2-1)1-5+10-10+5-1(1 \\
+2-\ 6+\ 6-2\quad \\
-\ 1+\ 3-3+1 \\
\hline
1-3+\ 3-\ 1+0+0
\end{array}$$

In this operation, $+2$ and -1, the terms in the divisor whose signs are changed, are multiplied by 1, the first term of the quotient, and the products are arranged in a diagonal line, as represented. By adding -5 and $+2$, the terms in the second column, we obtain -3, the second term in the quotient. Now multiply $+2$ and -1 by the second term of the quotient, and arrange the products, -6 and $+3$, in a diagonal line, as before. By adding $+10$, -6 and -1, the terms in the third column, we obtain $+3$, with which we can proceed as before. By supplying the literal parts of the quotient, we find that the required quotient is $a^3-3a^2x+3ax^2-x^3$.

From what has been said, we may derive the following

RULE.

I. *Divide the dividend and divisor by the co-efficient of the first term of the divisor, after having arranged the dividend and divisor with reference to the powers of the same letter. The first term of the quotient is the same as that of the dividend.*

II. *Change the signs of all the terms in the divisor except the first, and then multiply the terms with changed signs, by the first term of the quotient, and arrange the series of products thus obtained in a diagonal line passing downwards, so that the first term of the series may be directly under the second term of the dividend.*

III. *Find the sum of the terms in the second column for the second term of the quotient, and multiply the terms in the divisor with changed signs by the second term in the quotient, and place the series of products in a diagonal line, so that the first term of the series may be directly under the third term of the dividend.*

IV. *Find the sum of the terms in the third column for the third term of the quotient, with which proceed as before.*

In some examples, it will be observed that there are absent terms in the divisor or dividend, and in such cases, supply the places of absent terms with ciphers. For the purpose of illustrating this point, take the following example.

Divide $a^6-3a^4x^2+3a^2x^4-x^6$ by $a^3-3a^2x+3ax^2-x^3$. Here there are no terms in the dividend which contain a^5, a^3, a, x, x^3 and x^5. That is, the second, fourth, and sixth terms are absent, or their co-efficients are ciphers.

Operation.

$$
\begin{array}{r|l}
1+3-3+1) & 1+0-3+0+3+0-1 \\
 & +3+9+9+3 \\
 & -3-9-9-3 \\
 & +1+3+3+1 \\
\hline
 & 1+3+3+1+0+0+0
\end{array}
$$

Hence, $a^3+3a^2x+3ax^2+x^3$=the required quotient.

(207.) We can now transform any equation into another whose roots are less or greater than the roots of the given equa-

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tion by a given quantity, by the aid of *Synthetic Division*, and thus avoid much labor.

We will first take the equation $x^3-7x+7=0$, which we have already transformed by division. The roots in the transformed equation are to be less by unity than those in the proposed equation. The several divisions are exhibited in the following

Operation.

$$
\begin{array}{rll}
1+1)1+0-7+7 & & \\
1+1-6 & & \\
\hline
1+1-6+1 & \therefore & R_1=1 \\
1+2 & & \\
\hline
1+2-4 & \therefore & R_2=-4 \\
+1 & & \\
\hline
1+3 & \therefore & R_3=3
\end{array}
$$

In this example, the co-efficient of x^2 is 0. We see that the co-efficients of the terms in the quotient are 1, 1, -6, and the remainder is 1. The other quotients are obtained in the same manner. We may omit all the terms in the first column except the first, and write the changed term in the divisor at the right. The operation may then be written as follows:

$$
\begin{array}{rrrrll}
1+0 & -7 & +7(1 & & & \\
1 & 1 & -6 & & & \\
\hline
1 & -6 & 1 & & \therefore & R_1=1 \\
1 & 2 & & & & \\
\hline
2 & -4 & & & \therefore & R_2=-4 \\
1 & & & & & \\
\hline
3 & & & & \therefore & R_3=3
\end{array}
$$

Hence, $x^3+3x^2-4x+1=0$ is the transformed equation.

As another example, transform the equation $x^4-3x^3-15x^2+49x-12=0$ into another whose roots shall be less by 3 than those of the given equation.

Operation.

1	-3	-15	$+49$	$-12(3$	
	3	0	-45	12	
	0	-15	4	0	$\therefore R_1=0$
	3	9	-18		
	3	-6	-14		$\therefore R_2=-14$
	3	18			
	6	12			$\therefore R_2=12$
	3				
	9				$\therefore R_3=9$

Hence, the transformed equation is $x^4+9x^3+12x^2-14x=0$.

PROPOSITION I.

(208.) *If a_1, a_2, a_3 - - - - a_n represent the n roots of an equation of the nth degree, taken in the order of their magnitudes, so that a_2 is less than a_1, a_3 less than a_2, and so on; and if a series of numbers, b_1, b_2, b_3, b_4 - - - b_n, in which b_1 is greater than a_1, b_2 a number between a_1 and a_2, b_3 a number between a_2 and a_3, and so on, be substituted for the unknown quantity in the given equation, the results thus obtained will be alternately positive and negative.*

For, let $x^n+Ax^{n-1}+Bx^{n-2}$ - - - - $+Ox+P=0$ represent the given equation whose roots are a_1, a_2, a_3 - - - a_n. Then, by what has already been shown, this equation may be written

$$(x-a_1).(x-a_2).(x-a_3) \text{ - - - - - } (x-a_n)=0$$

If, in this last equation, we substitute b_1, b_2, b_3, - - - b_n in succession, we have the following series of results:

(1) $(b_1-a_1).(b_1-a_2).(b_1-a_3)$ - - - - $(b_1-a_n)=$a positive quantity.
(2) $(b_2-a_1).(b_2-a_2).(b_2-a_3)$ - - - - $(b_2-a_n)=$a negative quantity.
(3) $(b_3-a_1).(b_3-a_2).(b_3-a_3)$ - - - - $(b_3-a_n)=$a positive quantity.
(4) $(b_4-a_1).(b_4-a_2).(b_4-a_3)$ - - - - $(b_4-a_n)=$a negative quantity.
- -
(n) $(b_n-a_1).(b_n-a_2).(b_n-a_3)$ - - - - $(b_n-a_n)=$a negative or positive quantity according as n is even or odd.

In the first series, the product is positive, since all the factors are positive, b_1 being greater than any of the roots. In the second series, one of the factors b_2-a_1 is negative, and all the others are positive; hence the product is negative. By examining each series of factors, it may be seen that the several products are alternately positive and negative, which was the thing to be demonstrated.

Cor. 1. If two numbers be substituted for x, and the results obtained by these two substitutions have like signs, then there must be some *even* number of roots between the two numbers, or *no* root between the two numbers.

Cor. 2. If two numbers be substituted for x, and the results obtained have contrary signs, then there must be some *odd* number of roots between the two numbers.

Cor. 3. If any number p, and every number greater than p, gives a positive result, when substituted for x, then p is greater than the greatest root. Therefore, if the signs of the alternate terms in any equation be changed, and p and every number greater than p renders the result positive, then $-p$ is less than the least root of the given equation.

As an example, take the equation $x^3-7x+7=0$. If we substitute 1 for x we obtain for the result $+1$. If we substitute 2 for x, the result is also $+1$. Hence, by Cor. 1, there must be an *even* number of roots between 1 and 2, and as the equation has only *three* roots, there are two roots between 1 and 2; that is, each root is equal to 1 increased by a proper fraction. If we now transform the given equation into another whose roots are less by unity, we shall obtain for the transformed equation $x^3+3x^2-4x+1=0$. If we substitute in succession, for x, 0.1, 0.2, 0.3, 0.4, we shall find that there is one root between 0.3 and 0.4. Hence, the first two figures of one of the roots in the proposed equation are 1.3.

PROBLEM.

(210.) *Having given an equation of the nth degree, it is required to find another of the $(n-1)$th degree, such that the roots of the former shall be limits to the roots of the latter.*

Let $x^n+A_1x+A_2x^2$ - - - - - $A_{n-1}x+A_n=0$ be the given equation, whose roots are a_1, a_2, a_3 - - - a_n, taken in the order of their magnitude. If we transform this equation into another whose roots are less by r, we have the following operation:

$$\begin{array}{llcccc}
1+A_1 & +A_2 & \text{- - - -} & +A_{n-2} & +A_{n-1} & +A_n(r \\
\underline{\quad r\quad} & \underline{B_1r} & & \underline{B_{n-3}r} & \underline{B_{n-2}r} & \underline{B_{n-1}r} \\
\quad B_1 & B_2 & & B_{n-2} & B_{n-1} & B_n \\
\underline{\quad r\quad} & \underline{C_1r} & & \underline{C_{n-3}r} & \underline{C_{n-2}r} & \\
\quad C_1 & C_2 & & C_{n-2} & C_{n-1} &
\end{array}$$

In this transformation, we represent A_1+r by B_1, A_2+B_1r by B_2, and so on. The co-efficient of x in the transformed equation is C_{n-1}. By retracing the steps we have

$$\begin{aligned}
C_{n-1}&=A_{n-1}+rB_{n-2}+rC_{n-2} && (a)\\
&=A_{n-1}+r(A_{n-2}+rB_{n-3})+r(A_{n-2}+B_{n-3}r+C_{n-3}r) && (b)\\
&=A_{n-1}+2rA_{n-2}+2r^2B_{n-3}+r^2C_{n-3} && (c)\\
&=A_{n-1}+2rA_{n-2}+2r^2(A_{n-3}+rB_{n-4})+r^2(A_{n-3}+\\
&\qquad rB_{n-4}+rC_{n-4}) && (d)\\
&=A_{n-1}+2rA_{n-2}+3r^2A_{n-3}+3r^3B_{n-4}+r^3C_{n-4} && (e)\\
&\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
&=A_{n-1}+2rA_{n-2}+3r^2A_{n-3}+4r^3A_{n-4}\text{ - - - - - - - -}\\
&\qquad (n-1)r^{n-2}A_1+nr^{n-1}. && (f)
\end{aligned}$$

Or, $C_{n-1}=nr^{n-1}+(n-1)A_1r^{n-2}+(n-1)(n-2)A_2r^{n-3}$ - - -

$$+2A_{n-2}r+A_{n-1} \qquad (1)$$

Now, since the roots of the transformed equation are less by r than those of the given equation, they are a_1-r, a_2-r, a_3-r - - - - a_n-r.

Since C_{n-1} is the co-efficient of the first power of the unknown quantity in the transformed equation, it is, by a preceding proposition, equal to the sum of all the different products that can be formed from the n roots, taken $n-1$ at a time. Hence, we have

$$C_{n-1}=\left\{\begin{array}{lll} (r-a_1).(r-a_2)(r-a_3) \text{-----} & \text{to } (n-1) & \text{factors} \\ +(r-a_1).(r-a_2).(r-a_4) & \text{"} & \text{"} \\ +(r-a_1).(r-a_3).(r-a_4) & \text{"} & \text{"} \\ \quad\vdots \qquad \vdots \qquad \vdots & & \\ +(r-a_2).(r-a_3).(r-a_4) & \text{"} & \text{"} \end{array}\right\} \text{(A)}$$

If we substitute a_1 for r in the second member of equation (A), it is plain that all the terms will become zero which contain a_1, and the equation will become

$$C_{n-1}=(a_1-a_2).(a_1-a_3)(a_1-a_4) \text{ - - - - to } (n-1) \text{ terms.}$$

Since a_1 is the greatest root of the proposed equation, each of the factors in the second member of the last equation is positive, and therefore the value of C_{n-1} is *positive.* Again, if a_2 be substituted for r in equation (A), it will become

$$C_{n-1}=(a_2-a_1).(a_2-a_3).(a_2-a_4) \text{ - - - - to } (n-1) \text{ terms.}$$

In this equation, all the factors in the right-hand member are positive except the first, a_2-a_1, which is negative, a_1 being greater than a_2. Hence, this value of C_{n-1} is *negative.* By continuing this process of reasoning, it will be found that if $a_1, a_2, a_3, \text{- - - -} a_n$, be successively substituted for r in equation (A), the results will be alternately positive and negative. But, by a preceding proposition, if a series of numbers $a_1, a_2, a_3 \text{ - - - - } a_n$, be substituted for the unknown quantity in any equation, and the results are alternately positive and negative, the real roots of the equation are situated between these numbers, the numbers and the roots being arranged in the order of their magnitude.

But the value of C_{n-1} in equation (A) is equal to the value of C_{n-1} in equation (1). Therefore, if the series of numbers $a_1, a_2, a_3, a_4 \text{ - - - - } a_n$, be likewise substituted in equation (1) for r, the results will be alternately positive and negative. Hence, the real roots of equation (1) are situated between the roots of the given equation, and it is therefore the equation required.

If we change r into x in equation (1), it becomes $nx^{n-1}+$

$(n-1)A_1x^{n-2}+(n-1)(n-2)A_2x^{n-3}$ - - - - - $2A_{n-1}x+A_{n-1}=0$, by making $C_{n-1}=0$. This equation may be derived from the given equation by multiplying each term by the exponent of x in that term, and then diminishing the exponent of x in each term by a unit. This equation is the limiting equation.

COR. 1. If we make $a_1=a_2$, then $r-a_1$ will be a factor of each term in the right-hand member of equation (A), which is equal to the right-hand member of equation (1), which is the limiting equation. By a preceding proposition $x-a_1$ is an exact divisor of the given equation. Hence, *if any equation and its limiting equation have a common divisor, that equation must have equal roots.*

COR. 2. If $a_1=a_2=a_3$, then $(x-a_1)^2$ will be found in each series of factors in equation (A), and the given equation and the limiting equation will have a common divisor of the form $(x-a_1)^2$.

We can, then, always determine whether an equation has equal roots by ascertaining whether the proposed equation and its limiting equation have a common measure. As an example, find the equal roots of the equation $x^3-5x^2-8x+48=0$. The limiting equation is $3x^2-10x-8=0$. For finding the common measure of these two polynomials, we have the following

Operation.

$$
\begin{array}{l}
x^3-5x^2-8x+48 \\
3 \\
\hline
3x^2-10x-8)\,3x^3-15x^2-24x+144\,(x \\
\qquad 3x^3-10x^2-\ 8x \\
\hline
\qquad -\ 5x^2-16x+144 \\
\qquad -\ 3 \\
\hline
3x^2-10x-8)\,15x^2+48x-432\,(5 \\
\qquad 15x^2-50x-\ 40 \\
\hline
\qquad 98)\,98x-392 \\
\hline
\qquad x-4)\,3x^2-10x-8\,(3x+2 \\
\qquad 3x^2-12x \\
\hline
\qquad 2x-8 \\
\qquad 2x-8 \\
\hline
\qquad 0
\end{array}
$$

Therefore $x-4$ is the common measure required, and there are two roots equal to 4. If we divide the given equation by $x-4$, we obtain the quadratic equation,

$$x^2-x-12=0,$$

or, $$x^2-x=12;$$

whence, $$x=4, \text{ or } -3.$$

Therefore, the three roots of the given equation are 4, 4, -3.

The solutions of the following examples will test the pupil's knowledge of the principles which have been developed in this chapter, and at the same time they will serve to fix these principles in his mind.

EXAMPLES.

1. One root of the cubic equation $x^3-6x^2+11x-6=0$ is 1; find the quadratic which contains the other two roots, and employ synthetic division in the operation.

Ans. $x^2-5x+6=0$.

2. One root of the cubic equation $x^3+3x^2-16x+12=0$ is 1; find the remaining roots, and employ synthetic division in the operation. *Ans.* 2 and -6.

3. Two roots of the biquadratic equation $4x^4-14x^3-5x^2+31x+6=0$ are 2 and 3; find the quadratic equation which contains the remaining roots, and employ synthetic division in the operation. *Ans.* $4x^2+6x+1=0$.

4. Two roots of the biquadratic equation $x^4-6x^3+24x-16=0$ are 2 and -2; find the other two roots.

Ans. $3\pm\sqrt{5}$.

5. Transform the equation $x^4+2x^3+3x^2+4x-12340=0$ into another whose roots shall be less by 10 than the roots of the given equation. *Ans.* $x^4+12x^3+663x^2+4664x=0$.

6. Transform the equation $x^5+2x^3-6x^2-10x-8=0$ into another whose roots are less by 2 than the roots of the given equation. *Ans.* $x^5+10x^4+42x^3+86x^2+70x-4=0$.

7. Transform the equation $5x^4-12x^3+3x^2+4x-5=0$ into another whose roots shall be less by 2 than the roots of the given equation, and employ synthetic division.

Ans. $5x^4+28x^3+51x^2+32x-1=0$.

8. Transform the equation $x^3+x^2-10x+4=0$ into another whose roots shall be greater by 4 than the roots of the given equation. *Ans.*

9. Transform the equation $x^4-4x^3+6x^2-12=0$ into one whose roots shall be greater by 5 than the roots of the given equation. *Ans.*

10. Transform the equation $x^3-6x^2+9x-12=0$ into another whose roots shall be less than the roots by 6. *Ans.*

11. Form the equation whose roots are 1, 2 and -3.

Ans. $x^3-7x+6=0$.

12. Form the equation whose roots are $3+\sqrt{5}$, $3-\sqrt{5}$, and -6. *Ans.* $x^3-32x+24=0$.

13. One root of the equation $x^4-7x^3+15x^2-11x+2=0$ is $2+\sqrt{3}$; what are the other roots of the equation.

Ans. $2-\sqrt{3}$, 1, and 2.

14. The equation $x^3-15x^2+66x-80=0$, has two roots whose sum is 13; find all the roots. *Ans.* 8, 5, and 2.

15. The roots of the eqtuation $x^3-15x^2+66x-80=0$, have a common difference; determine them. *Ans.* 2, 5, and 8.

16. In the equation $x^3-6x^2+11x-6=0$, one root is double another; determine all the roots. *Ans.* 1, 2, and 3.

17. The product of two roots of the equation $x^4+x^3-62x^2-80x+1200=0$, is 30; it is required to determine the roots.

Ans. 5, 6, $-6+2\sqrt{-1}$, and $-6-2\sqrt{-1}$.

18. Determine the roots of the equation $x^3-17x^2+94x-168=0$, two of them being in the proportion of $2:3$.

Ans. 4, 6, 7.

19. One root of the equation $x^4-5x^3-x+5=0$ is 5; find the other roots. *Ans.* 1 and $\frac{1}{2}(-1\pm\sqrt{-3})$.

20. The equation $x^3-\frac{7}{2}x^2+\frac{7}{2}x-1=0$, has two roots of the form $a, \frac{1}{a}$; determine all the roots. *Ans.* 1, 2, and $\frac{1}{2}$.

21. Determine the roots of the equation $x^4-10x^3+35x^2-50x+24=0$, they being of the form $a+1$, $a-1$, $b+1$, $b-1$.

Ans. 1, 2, 3, and 4.

22. Solve the following equations, whose roots are in arithmetical progression:*

1. $x^3-6x^2-4x+24=0$. *Ans.* -2, 2, 6.
2. $x^3-9x^2+23x-16=0$.
3. $x^3-6x^2+11x-6=0$.
4. $x^3-3x^2+6x+8=0$.

23. Solve the following equations, whose roots are in geometrical progression:

1. $x^3-7x^2+14x-8=0$. *Ans.* 1, 2, 4.
2. $x^3-13x^2+39x-27=0$.
3. $x^3-14x^2+56x-64=0$.
4. $x^3-26x^2+156x-216=0$.

24. Determine whether the equation $x^4-9x^2+4x+12=0$ has equal roots, and find the roots of the equation.

Ans. 2, 2, -3, and -1.

---

* Note.—If the roots of the equation $x^n-px^{n-1}+qx^{n-2}-$&c.$=0$ be in arithmetical progression, very neat formulas may be obtained for finding the least root and the common difference of the roots.

25. Having given the equation $2x^4-12x^3+19x^2-6x+9=0$, determine whether it has equal roots. *Ans.* It has.

26. Find the equation whose roots are less by 1.7 than the roots of the equation $y^3-2y^2+3y-4=0$.

Ans. $y^3+3.1y^2+4.87y+0.233=0$.

27. One root of the equation $x^3-11x^2+37x-35=0$, being $3+\sqrt{2}$; find all the roots.

Ans. $3-\sqrt{2}$, $3+\sqrt{2}$, and 5.

28. Transform the equation $x^3-px^2+0.x-r$ into another whose roots are less by p than those of the proposed equation.

Ans. $x^3+2px^2+p^2x-r=0$.

STURM'S THEOREM.

(211.) *Sturm's Theorem* is a Theorem for finding the number of real and imaginary roots in any proposed equation. This was a problem of great difficulty, but the ingenious mathematician, M. Sturm, has given its solution.

THEOREM OF STURM.

Let $x^n+Ax^{n-1}+Bx^{n-2}$ - - - - $+Px+Q=0$ be any equation which has no *equal* roots, and let $nx^{n-1}+(n-1)Ax^{n-2}+(n-2)Bx^{n-3}$ - - - P be its first derived polynomial, formed by multiplying each term by the exponent of x in that term, and then diminishing the exponent of x by 1 in each term.

Divide the given equation by the derived polynomial, and continue the division till a remainder is obtained whose degree is less by unity than that of the divisor, and change the sign of this remainder. Now, divide the last divisor by the last remainder with its sign changed, and continue the division till a remainder is obtained whose degree is less by unity than that of the divisor, and

change the sign of this remainder. Proceed in this manner till a remainder is obtained which is independent of x.

If we denote the given equation by X, its derived polynomial by X_1, and the successive remainders, with their signs changed, by R_2, R_3, R_4, - - - - - R_{m+1}, we shall have a series of functions of x, the degrees of which regularly decrease, the last, or R_{m+1}, is independent of x. If we now substitute any number, m, for x in this series of functions, and note the number of variations of signs* in the results, and then substitute any other number, n, for x in the same series, and note the number of variations of signs in the results, then the exact number of real roots between m and n is found by taking the difference of the two numbers which express the number of variations of sign that arise from the two substitutions. This is the point to be established.

For the purpose of facilitating the demonstration of this theorem, we shall state the following lemmas.

LEMMA I.

Two consecutive functions cannot reduce to zero for the same value of x.

If we represent the successive quotients by Q_1, Q_2, Q_3 - - - - Q_m, we shall have, from the process which has been described for obtaining the series of functions, X, X_1, R_2, R_3 - - - - R_m, the following equations:

$$X=X_1Q_1-R_2 \qquad (1)$$
$$X_1=R_2Q_2-R_3 \qquad (2)$$
$$R_2=R_3Q_3-R_4 \qquad (3)$$
$$R_3=R_4Q_4-R_5 \qquad (4)$$
$$\vdots \qquad \vdots$$
$$R_{m-1}=R_mQ_m-R_{m+1} \qquad (m)$$

* NOTE.—By "the number of variations of signs" is meant the number of changes that can be made in passing from *plus* to *minus*, and from *minus* to *plus* in a series of signs. Thus if the series is $+--+-+++$, the number of variations is 4.

Let us suppose that $R_2=0$, and $R_3=0$. Hence, each of the terms R_2, R_3Q_3, in equation (3) must become 0, and therefore $R_4=0$. Since $R_3=0$, and $R_4=0$, we find from equation (4), that $R_5=0$, and finally, we shall find that $R_{m+1}=0$. But the equation $X=0$ has, by hypothesis, no equal roots, and consequently X and X_1 have no common measure, and therefore there must be a final remainder independent of x. Hence, we cannot suppose that two consecutive functions become zero for the same value of x.

LEMMA II.

If one of the derived functions reduces to zero for any particular value of x, the adjacent functions have contrary signs.

For, let us suppose that $R_3=0$, then equation (3) becomes $R_2 = -R_4$. But R_2 and R_4 are the adjacent functions, and as they have contrary signs, the lemma is true.

DEMONSTRATION.

Let p be less than the least root of the equations,

$$X=0,\ X_1=0,\ R_2=0\ \text{------}\ R_{m-1}=0,$$

and let q be greater than the greatest root of these equations. Now, so long as p remains less than the least root of any of these equations, the several factors into which each of the equations may be divided* must be negative, and therefore the signs of the results obtained by the substitutions of any numbers for x in the above equations, which are greater than p, and less than the least root of any of the equations, must be the same for each of these numbers.

Now, as p increases, it must pass successively over the roots of the given equation, and also over those of the derived equations. As p passes over one of the roots of the derived equations, that

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* NOTE.—It has been shown in Prop. III., page 259, that any equation may be resolved into as many factors as it has roots.

equation will vanish; but, by Lemma I., neither of the adjacent functions can vanish for the same value of x, and, by Lemma II., these two adjacent functions must have contrary signs. Hence, when one of the derived equations vanishes, the number of variations of signs is not altered.

Now, let p gradually increase till it becomes *very nearly* equal to one of the roots of the equation $X=0$, and let *this* value of p be substituted for x in the equations $X=0$, $X_1=0$, and the results will have *contrary* signs. Now as p increases till it becomes a *very little* greater than this root of the equation $X=0$, the sign of X must have changed, but the difference between these two values of p may be made so small, that in passing from that value of p which is a *little less* than a root of the equation $X=0$, to the one which is a *little greater* than this root, p could not have passed over any of the roots of the other functions, and consequently their signs remain unchanged. Before this small increase in the value of p the signs of X and X_1 were *contrary*,* and after

* NOTE.—Let it be observed that if we substitute for x in any equation a number a *very little less* than one of its roots, and also substitute the same number for x in its *limiting equation*, the two results will have *contrary* signs. For example, take the cubic equation $x^3+2x^2-6x+4=0$, and its limiting equation $3x^2+4x-6=0$, and represent the roots of the former by a_1, a_2, a_3, and the roots of the latter by b_1, b_2. Then the cubic equation may be written

$$(x-a_1).(x-a_2).(x-a_3)=0,$$

and the limiting equation may be written

$$(x-b_1).(x-b_2)=0$$

If the roots of the two equations be arranged in the order of their magnitude, regarding a_1 as being the least root, the arrangement will be a_1, b_1, a_2. b_2. a_3. If we substitute c, a number a *very little less* than a_2, for x in the two equations, they will become

$$(c-a_1).(c-a_2).(c-a_3)=\text{a } \textit{positive} \text{ quantity,}$$
and, $$(c-b_1).(c-b_2)=\text{a } \textit{negative} \text{ quantity.}$$

The first result is positive, since $(c-a_2)$ is a *negative* quantity, and $(c-a_3)$

it, they became *alike*. Hence, whenever p passes over a root of the equation $X=0$ there is a loss of *one* variation in sign. Suppose, that in substituting that value of p which is a very little less than a root of the equation $X=0$, for x in each of the equations, we obtain results which are affected with the following series of signs:

$$\begin{array}{cccccl} X & X_1 & R_2 & R_3 & R_4 & \\ - & + & - & - & + & \text{3 variations.} \end{array}$$

Now, if we increase p till it becomes a *very little* greater than this root, the sign of X will become changed, while the signs of the other functions will not be changed, and we shall have the following series of signs:

$$\begin{array}{cccccl} X & X_1 & R_2 & R_3 & R_4 & \\ + & + & - & - & + & \text{2 variations.} \end{array}$$

In substituting any number which is a very little less than a root of the equation $X=0$, we may have any order of signs in which the signs of X and X_1 are contrary. The last term, being independent of x, its sign is not changed by any of the substitutions.

We have seen that when p, by its continual increase, reaches and passes a root of the equation $X=0$, there is a loss of one variation. Suppose now that it increases from a number a very little greater than this root, till it reaches a number a very little less than that of the next root of this equation. During this increase of p it may pass over some of the roots of the other equations, thus changing the *order* of their signs, but the number of

---

is *negative*, a_3 being greater than a_2. The factor $(c-a_1)$ is *positive*, since c is *very nearly* equal to a_2, and a_1 is less than a_2. Hence, *two* of the factors being negative, and the other *positive*, the result is positive. The second result is negative, since $(c-b_1)$ is a positive quantity for the reason that a_2 is greater than b_1 and is *very nearly* equal to c. It is clear that the other factor, $(c-b_2)$ is negative. Hence, the product of the two factors is a *negative* quantity. If we take an equation of any other degree, and reason in a similar manner, we shall arrive at the same result.

variations is neither increased nor diminished. (Lemma II.) If we substitute this value of p, then, for x in each of the equations, we shall obtain the same number of variations of signs as we would obtain if we substituted any other value of p which is comprised between the limits of these two roots. But for this value of p, X and X_1 must have *contrary* signs, as has been shown in the note. Now let p increase till it becomes a *very little* greater than the next root of the equation $X=0$, and the sign of X will have changed, but the signs of the other functions will not have changed, for a reason already given. Hence, the signs of X and X_1 are now *alike*, and consequently another variation of sign has been lost as p passed another root of the equation $X=0$.

From what has now been shown, we know that if we substitute a number p which is less than a root of the equation $X=0$, for x in each of the equations $X=0$, $X_1=0$, $R_2=0$ - - - - - $R_{m-1}=0$, and note the number of variations of sign, and then substitute a number q which is greater than the next root of the equation $X=0$, for x in the same equations, and note the number of variations of signs, the difference in the number of variations is 2, which is the number of roots between p and q. In the same manner it may be shown, that if there are more than two roots between the limits of p and q, this number may always be determined by substituting p and q for x in each of the above equations, and then taking the difference of the two numbers which express the number of variations of signs corresponding to the two substitutions.

APPLICATION OF STURM'S THEOREM.

EXAMPLES.

1. Find the number and situation of the roots of the equation $x^3-x^2-2x+8=0$.

In this example we have

$$X=x^3-x^2-2x+1$$
$$X_1=3x^2-2x-2$$

For finding the other functions we have the following operations:

$$
\begin{array}{rl}
 & x^3-x^2-2x+1 \\
 & 3^{*} \\
3x^2-2x-2) & 3x^3-3x^2-6x+3(x-1 \\
 & 3x^3-2x^2-2x \\
 & \quad -x^2-4x+3 \\
 & \quad 3 \qquad \text{Multiply by 3 to avoid fractions.} \\
 & \quad -3x^2-12x+9 \\
 & \quad -3x^2+\ 2x+2 \\
 & \qquad 7)-14x+7 \\
 & \qquad -\ 2x+1 \quad \therefore R_2=2x-1
\end{array}
$$

$$
\begin{array}{rl}
 & 3x^2-2x-2 \\
 & 2 \\
2x-1) & 6x^2-4x-4(3x-1 \\
 & 6x^2-3x \\
 & \quad -\ x-4 \\
 & \qquad 2 \\
 & \quad -2x-8 \\
 & \quad -2x+1 \\
 & \qquad -9 \quad \therefore R_3=+9
\end{array}
$$

Hence, we have the following series of functions:

$$
\begin{aligned}
X&=x^3-x^2-2x+1\\
X_1&=3x^2-2x-2\\
R_2&=2x-1\\
R_3&=9
\end{aligned}
$$

Now, as all the real roots of any equation are comprised between the limits of $+\infty$ and $-\infty$, we can readily find the number of roots by substituting $+\infty$ and $-\infty$ for x in the first

* NOTE.—In the application of this Theorem, we may multiply or divide any of the quantities by a *positive* factor, but we cannot do the same with a *negative* factor, and the reason of this is plain.

terms of the several functions; for if x be equal to *infinity*, any power of x is greater than the sum of all the following terms in the function, and therefore the sign of the first term is the same as that of the whole function. Let m=the number of variations of sign which is obtained by substituting $+\infty$ for x, and n the number for $-\infty$; then $m-n$=the number of real roots in the equation.

If we substitute $+\infty$ for x in the above series of functions, we have the following series of signs

$$+ \quad + \quad + \quad + \quad \text{giving 0 variations.}$$

If we substitute $-\infty$ for x in the same series of functions, we have the following series of signs

$$- \quad + \quad - \quad + \quad \text{giving 3 variations.}$$

Therefore, $3-0=3$ is the whole number of real roots, and as the equation has only 3 roots, it follows that they are all real.

We will now find the situations of the roots. For this purpose we must employ narrower limits than $+\infty$ and $-\infty$. Let us commence at 0, in the series

$$-3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3, \ \&c.$$

and extend the limits in both directions.

SIGNS.

For $x=1$	we have	$- \ - \ + \ +$	giving 1 variation.
$x=2$	"	$+ \ + \ + \ +$	giving 0 variations.
$x=0$	"	$+ \ - \ - \ +$	giving 2 variations.
$x=-1$	"	$+ \ + \ - \ +$	giving 2 variations.
$x=-2$	"	$- \ + \ - \ +$	giving 3 variations.

It is unnecessary to proceed any farther, since, by inspecting the column of variations, we discover that there is one root between 1 and 2, one between 0 and 1, and one between -1 and -2. Hence, the initial figures of two of the roots are 1 and -1.

To find the initial figure of that root which is between 0 and 1, we might transform the given equation into another whose roots are less by 1 than those of the given equation, and then apply the theorem to the transformed equation.

2. Find the number and situation of the roots of the equation $x^3-4x^2-6x+8=0$.

In this example we find that the functions are

$$X=x^3-4x^2-6x+8$$
$$X_1=3x^2-8x-6$$
$$R_2=17x-12$$
$$R_3=514$$

Substitute $+\infty$ and $-\infty$ for x in the first terms of the several functions, and we have the following series of signs:

SIGNS.

For $x=\infty$ we have $++++$ giving 0 variation.
$x=-\infty$ " $-+-+$ giving 3 variations.

$\therefore\ m-n=3-0=3$, the number of real roots in the given equation.

For $x=0$	we have	$+--+$	giving	2	variations.
$x=1$	"	$--++$	"	1	"
$x=2$	"	$--++$	"	1	"
$x=3$	"	$--++$	"	1	"
$x=4$	"	$-+++$	"	1	"
$x=5$	"	$-+++$	"	1	"
$x=6$	"	$++++$	"	0	"
$x=-1$	"	$++-+$	"	2	"
$x=-2$	"	$-+-+$	"	3	"

By examining the column of variations, we discover that the three roots of the proposed cubic equation are between the limits of $+6$ and -2, and that there is one root between 0 and 1, one between 5 and 6, and one between -1 and -2. Hence, the initial figures of the root are 0, 5, and -1. It may be observed

that 1 very nearly satisfies the given equation, when substituted for x, and therefore the root, which is between 0 and 1, must be very nearly equal to 1. Hence, in order to determine the first decimal figure of this root, substitute 1 for x in the series of functions, then substitute 0.9, 0.8, 0.7, &c. for x in the same series, and continue the substitution till the situation of the root is determined.

Thus for $x=1$ we have $--++$ giving 1 variation.
$x=0.9$ " $+-++$ " 2 variations.

Hence, the initial figure of this root is 0.9.

3. Find the conditions that all the roots of the general cubic equation $x^3+ax^2+bx+c=0$, may be real.

In this example we have the following functions:

$$\left.\begin{aligned} X&=x^3+ax^2+bx+c\\ X_1&=3x^2+2ax+b\\ R_2&=2(a^2-3b)x+ab-9c\\ R_3&=-4a^3c+a^2b^2-18abc-4b^3-27c^2 \end{aligned}\right\}\quad\text{(A)}$$

In order that all the roots may be real, there must be no variations of sign when $+\infty$ is substituted for x in the several functions, and three variations of sign when $-\infty$ is substituted for x. Hence, we must have for the conditions required

$$(a^2-3b)>0 \qquad (1)$$
and
$$(a^2b^2-4a^3c-18abc-4b^3-27c^2)>0 \qquad (2)$$

When $a=0$, we have the following functions:

$$\left.\begin{aligned} X&=x^3+bx+c\\ X_1&=3x^2+b\\ R_2&=-2bx-3c\\ R_3&=-4b^3-27c^2 \end{aligned}\right\}\quad\text{(B)}$$

In order that all the roots of the equation $x^3+bx+c=0$ may be real, the first terms of the several functions must be positive, and therefore $-2bx$, and $-4b^3-27c^2$ must be positive. As $-27c^2$ is negative for all values of c, whether positive or negative,

b must be negative, in order that $-4b^3$ and $-2bx$ may be positive, and at the same time, $4b^3$ must be greater than $27c^2$, or $\left(\frac{b}{3}\right)^3 > \left(\frac{c}{2}\right)^2$.

Functions (A) and (B) may be used in applying Sturm's Theorem to particular examples in cubic equations. In order to obtain the several functions of x, we only substitute for a and b, their values in the given example.

4. Find the number and situation of the roots of the equation $x^3+11x^2-102x+181=0$.

The functions are

$$X = x^3+11x^2-102x+181$$
$$X_1 = 3x^2+22x-102$$
$$R_2 = 122x-393$$
$$R_3 = +$$

For $x=+\infty$,	we have	$++++$	giving	0 variations.
$x=-\infty$	"	$-+-+$	"	3 "

$\therefore\ m-n=3-0=3$, the number of real roots.

For $x=0$,	we have	$+--+$	giving	2 variations.
$x=1$	"	$+--+$	"	2 "
$x=2$	"	$+--+$	"	2 "
$x=3$	"	$+--+$	"	2 "
$x=4$	"	$++++$	"	0 "

Therefore two of the roots are between 3 and 4. In order to determine these roots within narrower limits, we will transform the given equation into another, whose roots are less by 3. The transformed equation is $x^3+20x^2-9x+1=0$, and for this equation we have the following functions:

$$X = x^3+20x^2-9x+1$$
$$X_1 = 3x^2+40x-9$$
$$R_2 = 122x-27$$
$$R_3 = +$$

For $x=0$	we have	$+--+$	giving	2 variations.
$x=0.1$	"	$+--+$	"	2 "
$x=0.2$	"	$+--+$	"	2 "
$x=0.3$	"	$++++$	"	0 "

Therefore there are two positive roots between 0.2 and 0.3, belonging to this transformed equation, and hence the given equation has two roots between 3.2 and 3.3. If we transform x^3+20x^2-9x+1 into an equation whose roots are less by 0.2, we shall obtain $x^3+20.6x^2-0.88x+0.008=0$ for the transformed equation. The functions of this equation are

$$X=x^3+20.6x^2-0.88x+0.008$$
$$X_1=3x^2+41.2x-0.88$$
$$R_2=122x-2.6$$
$$R_3=+$$

For $x=0$	we have	$+--+$	giving	2 variations.
$x=0.01$	"	$+--+$	"	2 "
$x=0.02$	"	$---+$	"	1 variation.
$x=0.03$	"	$++++$	"	0 "

Hence, this transformed equation has one root between 0.01 and 0.02, and one root between 0.02 and 0.03. Therefore two roots of the proposed equation are 3.21 and 3.22. Since the sum of the three roots of the equation is -11, the co-efficient of the second term taken with a contrary sign, the other root is $-11-3.21-3.22=-17.4$.

HORNER'S METHOD OF RESOLVING NUMERICAL EQUATIONS OF ALL DEGREES.

THERE are several methods for finding the roots of numerical equations of all degrees, but in point of simplicity and elegance, the method of W. G. Horner, of Bath, England, is superior to any that has yet been given. The initial figures of the roots of any equation may be found by Sturm's Theorem, and then the equation may be transformed into another whose roots are less than those of the proposed equation by one of these initial figures. The initial figures of the roots of this transformed equation may now be determined by Sturm's Theorem, and this process may be continued till a transformed equation of the form

$$x^n + Ax^{n-1} + Bx^{n-2} \text{ - - - - } Ox + P = 0$$

is obtained, in which x is a small quantity, as 0.08, for example. Now, as x is a small quantity in the above transformed equation, the higher powers of x must be smaller still, so that we shall obtain a near value of x, in the transformed equation, if we neglect the terms which contain these powers. By omitting these terms the transformed equation becomes

$$Ox + P = O,$$

$$\text{or,} \qquad x = -\frac{P}{O};$$

that is, the next figure of the root may be obtained by employing the last, or penultimate co-efficient, of each transformation, as a *trial divisor* of the absolute term, and with this figure of the root, we may proceed as before. Hence, by successive transformations, we may obtain the roots of a numerical equation to any required degree of exactness.

(212.) It is advisable to first find the positive roots of the proposed equation. To find the negative roots, change the signs of the alternate terms, and the positive roots of the resulting

equation, taken with a negative sign, will be the negative roots of the given equation.

EXAMPLES.

1. Find all the roots of the equation $x^3-3x-1=0$.

By Sturm's Theorem, the several functions are

$$\begin{aligned} X&=x^3-3x-1=0 \\ X_1&=3x^2-3 \\ R_2&=6x+3 \qquad\qquad (A) \\ R_3&=9 \end{aligned}$$

For $x=0$,	we have	$--++$	giving 1	variation.
$x=1$	"	$-0++$	" 1	"
$x=2$	"	$++++$	" 0	"

Hence, there is one root between 1 and 2, and the initial figure of this root is therefore 1. In order to obtain the first decimal figure of this root, we must transform the preceding functions into others, whose roots are less by unity. Thus, for the function X, we have the following

Operation.

$$\begin{array}{rrrl} 1+0 & -\;3 & -\;1 & (1 \\ 1 & 1 & -2 & \\ \hline 1 & -2 & -3 & \\ 1 & 2 & & \\ \hline 2 & 0 & & \\ 1 & & & \\ \hline 3 & & & \end{array}$$

And transforming the others in the same way, we have the following series of transformed functions:

$$\begin{aligned} X&=x^3+3x^2-3 \\ X_1&=3x^2+6x \\ R_1&=6x+9 \qquad\qquad (B) \\ R_2&=9 \end{aligned}$$

For $x=1$, we have $++++$ giving 0 variation.
$x=0.9$ " $++++$ " 0 "
$x=0.8$ " $-+++$ " 1 "

Hence, one root of the transformed equation is between 0.8 and 0.9, and therefore the initial figure of this root, or the first decimal figure of the corresponding root in the proposed equation, is 0.8. To obtain the next decimal figure in the root, we must transform functions (B) into others, whose roots are less by 0.8. For transforming the first of these functions, we have the following

Operation.

1	3	0	−3	(0.8
	0.8	3.04	2.432	
	3.8	3.04	−0.568	
	0.8	3.68		
	4.6	6.72		
	0.8			
	5.4			

And transforming the others in the same way we have

$$
\left.\begin{array}{l}
X=x^3+5.4x^2+6.72x-0.568 \\
X_1=3x^2+10.8+6.72 \\
R_2=6x+13.8 \\
R_3=9
\end{array}\right\} \quad (C)
$$

If we were to apply Sturm's Theorem to this series of functions, we should find that the function X had one root between 0.07 and 0.08, and therefore the next decimal figure of the corresponding root in the proposed equation is 0.07, and the root itself, as thus far determined, is 1.87. Now, according to what has been shown, the value of x in the first equation of functions (C), may be nearly found by dividing the absolute term, 0.568 by 6.72, the co-efficient of x; and this division gives for this value of x 0.08, nearly. Now, if we transform this equation into another whose roots are less by 0.08, we obtain for the transformed equation

$$x^3+5.64x^2+7.6032x+0.004672=0.$$

As all the terms of this equation are *positive*, it follows that all its roots are *negative*. Hence, in diminishing a positive root of the equation,

$$x^3+5.4x^2+6.72x-0.568=0,$$

by 0.08, we must have rendered it *negative*, and hence this positive root is *less* than 0.08. It will be observed that the last transformation caused the absolute term to change its sign. Now, it is plain that the absolute term cannot change its sign, unless the co-efficient of x changes its sign at the same time. In obtaining, then, a positive root of an equation, we must recollect, *that whenever a figure of the root is obtained by means of the trial divisor, which causes the sign of the absolute term to change, but not that of the co-efficient of x, this figure must be diminished, till one is obtained which does not cause such a change of signs.*

Since the first figure of the root of the equation

$$x^3+5.4x^2+6.72x-0.568=0,$$

is 0.07, we will, in order to obtain the next figure in the root, transform it into another, whose roots are less by 0.07. For this transformation, we have the following

Operation.

1	5.4	6.72	−0.568	(.07
	0.07	0.3829	0.497203	
	5.47	7.1029	−0.070797	
	0.07	0.3878		
	5.54	7.4907		
	0.07			
	5.61			

Hence, the transformed equation is

$$x^3+5.61x^2+7.4907x-0.070797=0$$

$$\therefore\ x=\frac{0.070797}{7.4907}=0.009, \text{ nearly.}$$

We can continue this process of successive transformations, till

the root of x is obtained to any required degree of exactness. For the sake of brevity, these transformations may be condensed into one operation, as follows:

Operation.

	Col. I.	Col. II.	Col. III.	Root.
1	0	− 3	− 1	1.87938
	1	1	− 2	
	1	− 2	−*3	
	1	2	2.432	
	2	*0	−*0.568	
	1	3.04	0.497203	
	*3.8	3.04	−*0.070797	
	0.8	3.68	0.067871439	
	4.6	*6.72	−*0.002925561	
	0.8	0.3829	0.002278084257	
	*5.47	7.1029	−*0.000647476743	
	0.07	0.3878		
	5.54	*7.4907		
	0.07	0.050571		
	*5.619	7.541271		
	0.009	0.050652		
	5.628	*7.591923		
	0.009	0.00169119		
	*5.6373	7.59361419		
	0.0003	0.00169128		
	5.6376	*7.59560547		
	0.0003			
	*5.63798			

By the application of Sturm's Theorem, we find that the initial

* The co-efficients of the successive transformed equations are indicated by asterisks.

figure of one of the negative roots is —0.3, and for the successive transformations, we have the following

Operation.

	Col. I.	Col. II.	Col. III.	Root.
1	0	— 3	— 1	—0.3473
	— 0.3	0.09	0.873	
	— 0.3	— 2.91	—*0.127	
	— 0.6	0.18	0.107696	
	— 0.3	—*2.73	—*0.019304	
	—*0.94	0.0376	0.018522077	
	— 0.04	— 2.6924	—*0.000781923	
	— 0.98	0.0392		
	— 0.04	—*2.6532		
	—*1.027	0.007189		
	— 0.007	— 2.646011		
	— 0.034	0.007238		
	— 0.007	—*2.638373		
	—*0.0413			

Since the algebraical sum of the three roots is equal to the co-efficient of x^2, the other root is equal to $0-(1.8793-0.3473)=$ -1.5320.

(213.) Let it be observed that each successive figure in the decimal part of the root extends Col. I. *one* decimal place, Col. II. *two*, and Col. III. *three* decimal places. We may also observe, that the co-efficients of x^2 and x in the successive transformed equations change in value very slowly after the third or fourth transformation. These co-efficients are indicated by the asterisks in each column. In the operation for finding the positive value of x, in the preceding example, it will be noticed that the co-efficient of x in the last transformed equation is 7.59 nearly, and that it is 7.59 nearly, in the last transformed equation but one. Hence, after three or four decimals of the root have been found, we may find several of the remaining decimals of the root by dividing the

absolute term in the last transformed equation, by the co-efficient of x in that equation. Now, since the last decimal figure in the root extended this absolute term *three* decimal places, and the co-efficient of x only *two* decimal places, we may omit, in performing this division, a number of decimals in the dividend, and a number less by one, in the divisor. The accuracy of the result will not be much affected. The division may be performed by the *abridged method** for the division of decimals.

For finding three or four additional figures of the positive root of x, in the last example, we have the following

Operation.

```
7.59560)0.0006474767(0.000085243
          6076480
          -------
           398287
           379780
           ------
            18507
            15191
            -----
             3316
             3038
             ----
              278
              227
              ---
```

Hence, $x=1.879385243$. The last figure should be 2 instead of 3.

2. Find the roots of the equation

$$x^3+11x^2-102x+181=0.$$

This equation is one of the examples in the application of Sturm's Theorem, and we then found that the roots were nearly 3.21, 3.22, and -17.

---

* See Perkins's Higher Arithmetic, and Davies' University Arithmetic.

Operation.

	Col. I.	Col. II.	Col. III.	
1	11	−102	181	3.213127
	3	42	−180	
	14	− 60	1	
	3	51	−0.992	
	17	− 9	0.008	
	3	4.04	−0.006739	
	20.2	−4.96	0.001261	
	0.2	4.08	−0.001217403	
	20.4	−0.88	0.000043597	
	0.2	0.2061		
	20.61	−0.6739		
	0.01	0.2062		
	20.62	−0.4677		
	0.01	0.061899		
	20.633	−0.405801		
	0.003	0.061908		
	20.636	−0.343893		
	0.003			
	20.639			

```
343893)0.000043597(0.000127
            34389
             9208
             6878
             2330
             2406
```

The other roots may be found in the same way. They ar. 3.229521, and −17.44265.

3. Find a root of the equation

$$x^4+x^3+x^2+3x-100=0.$$

By Sturm's Theorem, we have the following functions:

$$X=x^4+x^3+x^2+3x-100$$
$$X_1=4x^3+3x^2+2x+3$$
$$R_2=-5x^2-34x+1603$$
$$R_3=-1132x+6059$$
$$R_4=-$$

If in these functions, we substitute -3 for x, and then -4, and note the number of variations of sign in each case, we shall find that there is a loss of one variation of sign, and hence the initial figure of one of the roots is -3. To find this root we have the following

Operation.

−1	+1	−3	−100(3.43
3	6	21	54
2	7	18	46
3	15	66	41.6896
5	22	84	4.3104
3	24	20.224	3.84456501
8	46	104.224	.46583499
3	4.56	22.112	
11.4	50.56	126.336	
0.4	4.72	1.816167	
11.8	55.28	128.152167	
0.4	4.88		
12.2	60.16		
0.4	0.3789		
12.63	60.5389		

The student may find three or four more decimals of this root. The root is -3.433577, and the other root is 2.8028. Two of the roots are imaginary.

4. Find one root of the equation

$$x^3-22x-24=0.$$

Ans. $x=5.16227$.

5. Find a root of the equation

$$x^3+x^2-500=0.$$

Ans. $x=7.617279.$

6. Find a root of the equation

$$x^3+x^2+x-100=0.$$

Ans. $x=4.264429.$

7. Find a root of the equation

$$2x^3+3x^2-4x-10=0.$$

Ans. $x=1.624819.$

8. Find a root of the equation

$$x^4-12x^2+12x-3=0.$$

Ans. $x=2.85808.$

9. Find a root of the equation

$$x^3-9=0.$$

Ans. $x=2.0800.$

10. Find a root of the equation

$$x^3+2x^2+3x=13089030.$$

Ans. $x=235.$

11. Find a root of the equation

$$x^3-17x^2+54x=350.$$

Ans. $x=14.95406.$

12. Find a root of the equation

$$x^3-18\frac{1}{12}x+29\frac{53}{108}=0.$$

Ans. $x=2.333$, or $2\frac{1}{3}$.

APPENDIX.

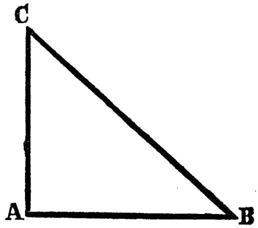
APPLICATION OF ALGEBRA TO GEOMETRY.*

(214.) No general rules can be given for the solution of geometrical problems, but if the student is familiar with the principles of algebra and geometry, he will be enabled, by practice, to solve such problems as follow, without much difficulty.

PROBLEM I.

Having given, in a right-angled triangle, the sum of the base and perpendicular, and the hypothenuse, it is required to determine the base and perpendicular.

Let A B C represent the right-angled triangle, right-angled at A. We shall represent quantities which are known, by the first letters of the alphabet, and those which are not known, by the last letters of the alphabet.



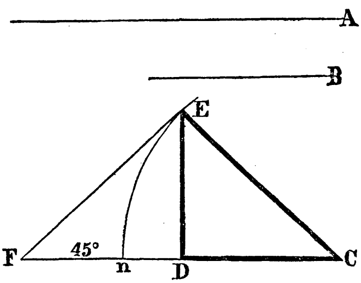
Let x=A C, the perpendicular,
y=A B. the base,
$x+y=a$, the sum of the base and perpendicular,
and c=C B, the hypothenuse.

* NOTE.—The Geometry referred to is Davies' Legendre, new edition.

Then, $x+y=a$, (1)
And, $x^2+y^2=c^2$. (2) (Geom. Bk. IV., Prop. XI.)
Eq. (1) squared, gives $x^2+2xy+y^2=a^2$ (3)
Eq. (3)—Eq. (2), gives $2xy=a^2-c^2$ (4)
Eq. (2)—Eq. (4) gives $x^2-2xy+y^2=2c^2-a^2$ (5)
The sq. root of eq. (5) gives $x-y=\pm\sqrt{2c^2-a^2}$ (6)
Eq. (6)+eq. (1) gives $2x=a\pm\sqrt{2c^2-a^2}$ (7)
Eq. (1)—Eq. (6) gives $2y=a\mp\sqrt{2c^2-a^2}$ (8)
$\therefore$ $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{2c^2-a^2}$ (9)
And, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{2c^2-a^2}$ (10)

If we let $a=7$, $c=5$, and then substitute these values for a and c in equations (9) and (10), we shall find that $x=4$, or 3, and that $y=3$, or 4.

We will now give the *geometrical* solution of this problem, in order that the student may compare the two methods of solution.

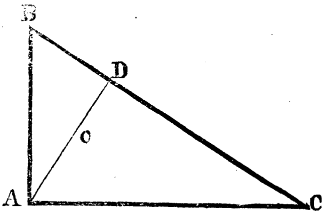


Let the line A represent the sum of the base and perpendicular, and let the line B represent the given hypothenuse. Take F C equal to A, and from the point C, as a centre, with a radius equal to B, describe the indefinite arc E n. From the point F draw the line F E, making the angle E F D equal to half a right-angle, and intersecting the arc E n at E. From the point of intersection, E, draw E D perpendicular to F C, and draw E C. E D C is the triangle required.

For, since the angle E F C is equal to half a right-angle, the other acute angle F E D, in the right-angled triangle E F D, must also be half a right-angle. Hence, the triangle E F D is isosceles, and E D equals F D. Therefore, E D + D C = F C = A. By construction, C E = B, the given hypothenuse. Hence, the triangle E D C is the one required.

PROBLEM II.

In a right-angled triangle, having given the perimeter, and the perpendicular let fall from the right-angle on the hypothenuse; to determine its sides.



Let A B C represent the required triangle, right-angled at A, and A D the perpendicular let fall on the hypothenuse.

Let $x+y=$ A C, $x-y=$ A B, $p=$ the perimeter, Whence $p-2x=$ B C.

and $c=$ the perpendicular A D.

$(x+y)^2+(x-y)^2=(p-2x)^2$ (1) (Geom. Bk. IV., Prop. XI.)

$x^2-y^2=cp-2cx$ (2)

Equation (2) is obtained by finding two expressions for the area of the triangle B A C, and then equating them. Thus,

$(x+y).(x-y)=x^2-y^2=$ double the area of the triangle A B C,
And $c\times(p-2x)=cp-2cx=$ " " "

By expanding, and uniting terms in equation (1), we have

$$2x^2+2y^2=p^2-4px+4x^2 \quad (3)$$

By transposing, $-2x^2+2y=p^2-4px$ (4)

Eq. (2)$\times$2, gives $2x^2-2y^2=2cp-4cx$ (5)

Eq. (4)+Eq. (5) gives $0=p^2+2cp-4px-4cx$ (6)

Transposing, $4cx+4px=p^2+2cp$;

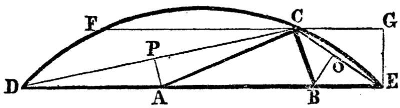
Whence, $x=\dfrac{p^2+2cp}{4c+4p}=b$, by putting b for this value of x. Substitute the value of x for x in equation (5), and it becomes

$$2b^2-2y^2=2cp-4bc, \quad (7)$$

Or, $y^2=2bc-cp+b^2$, (8)

Whence, $y=\pm\sqrt{2bc-cp+b^2}$. (9)

GEOMETRICAL SOLUTION.



Upon D E, the given perimeter, describe a segment such that every angle inscribed in it may be equal to 135°. Draw E G perpendicular to D E, and equal to the given perpendicular. Draw G F parallel to D E, intersecting the arc D C E in the points C and F. From one of these points, as C, draw the chords D C and C E, and bisect these chords by the perpendiculars A P and O B, and from the points A and B where these perpendiculars meet the line D E, draw A C and C B. A C B is the triangle required.

For, since D P=P C, by construction, and A P is common, the two right-angled triangles, A P D and A P C, are equal. (Geom., Book I., prop. V.) Hence, A D=A C. In a similar manner, it may be shown, that C B=B E. Therefore, the perimeter of the triangle A C B is equal to D E, the given perimeter.

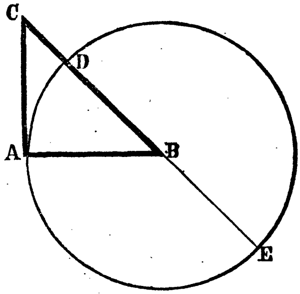
Further, the angle C A B equals the sum of the angles A D C, and A C D, or twice the angle A D C. (Geom., Book I., prop. XXV., Cor. 6.) In the same manner, it may be shown, that the angle A B C is equal to twice the angle A E C. The angle C D E is measured by one half of the arc E C. (Geom., Book III., prop. XVIII.) Hence, the angle C A B is measured by the whole arc C E. For a like reason, the angle C B A is measured by the arc D C. Hence, the two angles A B C and C A B are together measured by the arc D C E. But, by construction,* D C E is an arc of 90°. Hence, the sum of the two angles C A B and A B C is equal to one right-angle, since an arc of 90° is the measure of a right-angle. Therefore, the angle A C B is a right-angle, (Geom., Book I., prop. XXV.) As the perpendicular let fall

* By construction, the angle D C E=135°. Hence, the arc which measures it must be an arc of $2\times135°=270°$, and as this arc, together with the arc D C E, make up the whole circumference, or an arc of 360°, it follows that D C E is an arc of 90°.

from the right-angle A C B upon the hypothenuse A B, is obviously equal to E G, which, by construction, is equal to the given perpendicular, it follows that the triangle A B C answers all the conditions of the problem, and it is, therefore, the triangle required.

PROBLEM III.

Having given the base and perpendicular of a right-angled triangle, it is required to determine the hypothenuse.



Let A B C represent the right-angle triangle, right-angled at A. From the point B, as a centre, with a radius A B, describe the circumference of a circle, intersecting the hypothenuse C B in D. A C is a tangent to the circumference. (Geom., Book III., prop. IX.)

Let $AC = a$,

$AB = b$,

and, $CB = x$; whence, $CD = CB - DB = CB - AB = x - b$,

$CE = CB + BE = CB + AB = x + b$.

$CE : AC :: AC : CD$, (Geom., Book IV., prop. XXX.)

Or, $x + b : a :: a : x - b$,

Or, $$x^2 - b^2 = a^2; \quad (1)$$

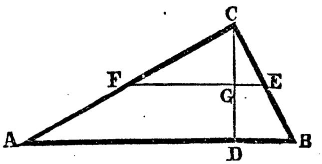
Whence, $$x^2 = a^2 + b^2, \quad (2)$$

And, $$x = \sqrt{a^2 + b^2}, \quad (3)$$

From equation (2), we see that *the square described on the hypothenuse of a right-angled triangle is equal to the sum of the squares described upon the base and perpendicular*, a theorem which is, perhaps, the most important of any in elementary geometry.

PROBLEM IV.

Having given the base and perpendicular of a plane triangle, it is required to bisect it by a line drawn parallel to the base.



Let A B C represent the triangle, of which the base A B, and the perpendicular C D, are given.

Let $AB=b$, $CD=p$, and $CG=x$. The line F E is drawn parallel to the base. By similar triangles, we have

$$CD:AB::CG:FE,$$

$$\text{Or,}\quad p:b::x:FE;\quad \therefore\ FE=\frac{bx}{p}.$$

$\frac{1}{2}bp=$ the area of triangle A B C,

and $\frac{bx^2}{2p}=$ the area of triangle C F E. (Geom., Book IV., prop. VI.)

Therefore, by the problem, we have

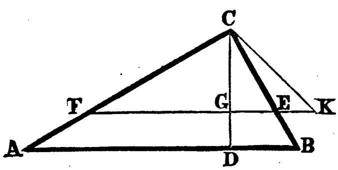
$$\frac{bx^2}{p}=\frac{1}{2}bp; \qquad (1)$$

$$\text{Whence,}\quad x=\sqrt{\tfrac{1}{2}p^2}=\frac{p}{2}\sqrt{2}. \qquad (2)$$

Therefore, if we take the line C G equal to $\frac{p}{2}\sqrt{2}$, and draw, through the point G, F E parallel to the base A B, the triangle A B C is divided into two equal parts.

The form which the value of x assumes suggests the following

GEOMETRICAL SOLUTION.



Draw C D perpendicular to the base A B, and then draw C K equal to C D, and making with C D an angle of 45°. Draw K F parallel to A B, and the triangle A C B is divided in the manner required, that is, the triangle F C E is equal to one half of the triangle A B C.

For, since the angle G C K=45°, by construction, it follows that the angle C K G must also equal 45°. Hence, the triangle G C K is isosceles, and C G=G K. Hence, $(CK)^2=(CG)^2+(GK)^2=2(GC)^2=(CD)^2$. By similar triangles, we have

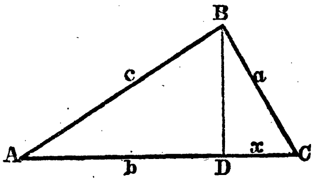
$$ABC : FCE :: (BC)^2 : (CE)^2 :: (CD)^2 : (GC)^2 :: 2 : 1.$$

Hence, the triangle F E C equals one half of the triangle A B C, which was the thing to be shown.

PROBLEM V.

Having given the three sides of a plane triangle, it is required to determine its area.

Let A B C represent the triangle, whose sides are A B $=c$, A C$=b$, and B C$=a$. Let fall the perpendicular B D, and let D C$=x$.



$$c^2=a^2+b^2-2bx\,;\ \text{(Geom., Book IV., prop. XII.)}$$

Whence, $x=\dfrac{a^2+b^2-c^2}{2b}$

$$BD=\sqrt{a^2-x^2}=\left\{a^2-\frac{(a^2+b^2-c^2)^2}{4b^2}\right\}^{\frac{1}{2}}=$$

$$\left\{\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4b^2}\right\}^{\frac{1}{2}}=\frac{[(a^2+2ab+b^2-c^2).(c^2-(a^2-2ab+b^2)]^{\frac{1}{2}}}{2b}$$

$$=\frac{[((a+b)^2-c^2).(c^2-(a-b)^2)]^{\frac{1}{2}}}{2b}=$$

$$\frac{[(a+b+c)(a+b-c)(c+a-b)(a-a+b)]^{\frac{1}{2}}}{2b}$$

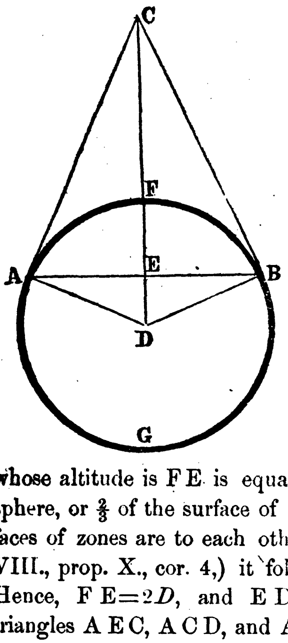
If we multiply the value of B D in its last form, by one half of A C, or $\frac{1}{2}b$, the product will be the area of the triangle A B C, and calling this area S, we have

$$S=\tfrac{1}{4}[(a+b+c).(a+b-c).(c+a-b).(c-a+b)]^{\frac{1}{2}}$$

The several changes in the form of the value of B D were made by the application of the theorem, that the difference of the squares of two quantities is equal to the product of their sum and difference. The quantities within the parentheses were regarded as being single quantities.

PROBLEM VI.

How high above the surface of the earth must a person be raised to see one-third of its surface?



Let A F B G represent a plane section of the earth made by a plane passing through its centre D. Let C represent the required point of elevation. Draw C D, and the two tangents A C and B C. A and B will be points in that circumference on the surface of the earth, which bounds the person's vision. Draw A D, B D, and A B.

Let x=FC the required altitude, and 3D=the radius of the earth.

Since the surface of the zone whose altitude is F E is equal to $\frac{1}{3}$ of the surface of the whole sphere, or $\frac{2}{3}$ of the surface of the hemi-sphere, and since the surfaces of zones are to each other as their altitudes, (Geom., Book VIII., prop. X., cor. 4,) it follows that D F is to F E as 3 : 2. Hence, F E$=2D$, and E D$=D$. Now, in the right-angled triangles A E C, A C D, and A E D, we have

$$(\text{A E})^2=(\text{A D})^2-(\text{E D})^2=9D^2-D^2=8D^2$$
$$(\text{A C})^2=(\text{A E})^2+(\text{C E})^2=8D^2+(x+2D)^2=12D^2+4Dx+x^2,$$
And $$(\text{A C})^2=(\text{C D})^2-(\text{A D})^2=(x+3D)^2-9D^2=x^2+6Dx.$$

Hence, $x^2+4Dx+12D^2=x^2+6Dx;$

Whence, $x=6D.$

Therefore, the person must be elevated to the height of the diameter of the earth, in order to see one-third of its surface. In this solution we have made no allowance for the atmospheric refraction of light, and we have assumed that the earth is a sphere.

PROBLEM VII.

Having given the area of a rectangle inscribed in a triangle, it is required to determine the triangle.

PROBLEM VIII.

Having given the perimeter of a right-angled triangle, and the radius of its inscribed circle, it is required to determine the triangle.

PROBLEM IX.

Having given the hypothenuse of a right-angled triangle, and the side of the inscribed square, it is required to determine the triangle.

PROBLEM X.

Having given the side of an inscribed square, in a right-angled triangle, and the radius of its inscribed circle, it is required to determine the triangle.

PROBLEM XI.

Find the area of a regular octagon, whose side is a.

PROBLEM XII.

Find the area of a regular hexagon, whose side is a.

PROBLEM XIII.

Find the area of a regular polygon of twelve sides, each side being equal to a.

PROBLEM XIV.

Determine the radii of three equal circles, described in a given circle, which are tangent to each other, and to the given circle.

PROBLEM XV.

The distances from a point P to two parallels, are denoted by a and b. It is required to draw a line through the point P intersecting the two parallels, so that the part of the line between the two parallels may be equal to a given line c.

PROBLEM XVI.

Draw a circle through to given points, and tangent to a given line.

PROBLEM XVII.

Four equal balls, $2a$ inches in diameter, are placed so as to form a triangular pyramid, and they are tangent to each other. Find the altitude of this pyramid.

PROBLEM XVIII.

If R and r are the radii of two spheres inscribed in a cone, so that the greater may touch the less, and also the base of the cone: then will the solidity of the cone be $\dfrac{2\pi R^5}{3r(R-r)}$

THE FARM AND THE FIRESIDE;

OR,

THE ROMANCE OF AGRICULTURE,

BEING

HALF HOURS OF LIFE IN THE COUNTRY, FOR RAINY DAYS AND WINTER EVENINGS.

BY REV. JOHN L. BLAKE, D.D.

AUTHOR OF FARMER'S EVERY-DAY BOOK; THE FARMER AT HOME; AND A GENERAL BIOGRAPHICAL DICTIONARY.

COMMENDATIONS OF THE PERIODICAL PRESS.

From the Ohio Farmer.

DR. BLAKE is justly regarded as one of the best agricultural writers in the country, and the work before us is one of the most interesting productions of his pen. Its peculiar merit, as a work for the fireside, consists in the variety of its topics, its plain and simple, yet attractive style, its fine engravings, and the interesting romance which the author has thrown around Rural and Agricultural Life. In this respect, "The Farm and the Fireside" is a work well adapted to the youthful mind. We hope it may be extensively read, as it cannot fail to improve the taste and promote inquiry in the most useful and practical of all departments of science.

From the New-York Evangelist.

The aim of the author has been to throw over labor, home and agricultural life, their true dignity and charm; to introduce the farmer to the delights and privileges of his lot; to embellish the cares of toil with those kindly sentiments so naturally associated with the country and its employments. It is a pleasant book—one that will enliven the

fireside, elevate and purify the thoughts, and, at the same time, impart a great deal of valuable agricultural knowledge. We know not how the natural trains of thought of the farmer could be more aptly met or more safely and agreeably led, than they are by these brief and varied discussions. The range is as wide as life itself—morals, religion business, recreation, education, home, wife and daughters—every relation and duty is touched upon, genially and instinctively.

From the New-York Tribune.

We have here another highly instructive and entertaining volume from an author, who had laid the community under large obligations by the enterprise and tact with which he has so frequently catered to the popular taste for descriptions of rural life. Its contents are of a very miscellaneous character, embracing sketches of natural history, accounts of successful farming operations, anecdotes of distinguished characters, singular personal reminiscences, pithy moral reflections, and numerous picturesof household life in the country. No family can add this volume to their collection of books without increasing their sources of pleasure and profit.

From the Northern Christian Advocate.

The venerable author of this work is entitled to the warmest thanks of the public for his numerous and valuable contributions to our literature. He is truly an American classic. We have been conversant with his writings for the last twenty years, and have always found them both useful and entertaining in a high degree. His writings on Agriculture contain much real science, with numerous illustrative incidents, anecdotes, and aphorisms, all in the most lively and pleasing manner. By this means the dry details of farming business are made to possess all the interest of romance. The style is clear, easy, and dignified; the matter instructive, philosophical, and persuasive. This work is an eloquent plea for the noble and independent pursuit of Agriculture.

From the National Magazine.

We return our thanks for the new volume of Dr. Blake, "The Farm and the Fireside, or the Romance of Agriculture, being Half Hours and

Sketches of Life in the Country," a charming title, certainly, and one that smacks of the man as well as of the country. Eschewing the dryness of scientific forms and erudite details, the author presents detached, but most entertaining, and often very suggestive articles on a great variety of topics—from the "Wild Goose" to "Conscience in the Cow,"—from the "Value of Lawyers in a Community" to the "Objections to early Marriages." The book is, in fine, quite unique, and just such a one as the farmer would like to pore over at his fireside on long winter evenings.

From the New-York Recorder.

"The Farm and the Fireside," is a most interesting and valuable work, being a series of Sketches relating to Agriculture and the numerous kindred arts and sciences, interspersed with miscellaneous moral instruction, adapted to the life of the farmer.

From the Germantown Telegraph.

We have looked through this work and read some of the "Sketches," and feel a degree of satisfaction in saying that it possesses decided merit, and will commend itself, wherever known, as a volume of much social interest and entertainment. The sketches comprise "Country Life" generally—some of them are just sufficiently touched with romance to give them additional zest; while others are purely practical, and relate to the farmer's pursuit. We regard it as a valuable book, and are sorry our limits will not admit of bestowing upon it such a notice as it really deserves.

From Harper's New Monthly Magazine.

This work is a collection of miscellaneous sketches on the Romance of Agriculture and Rural Life. Matters of fact, however, are not excluded from the volume, which is well adapted for reading in the snatches of leisure enjoyed at the farmer's fireside.

From the True Democrat.

Dr. Blake's publications are all of a high order, and are doing a most important work towards refining the taste, improving the intellect, and

rendering attractive the various branches of Agricultural science. Indeed we know no author who has so successfully blended the romantic, the rural and beautiful with the poetical, the useful, and true, as has Dr. Blake. This is a peculiar feature of all his works His style is plain, simple, and perspicuous; and, with unusual tact and judgment, he so manages to insinuate himself upon you, that you are at once amused, delighted, and instructed with the subject he is discussing. In this respect he relieves the study of agricultural science from the abstruseness of technical science, and thus renders himself easily comprehended by all classes of readers.

From the New-York Evening Post.

The author's object is to improve the soil through the mind—not so much to place in the hands of farmers the best methods of raising large crops—for these he refers them to Leibig's Agricultural Chemistry, and to treatises of the like description—but to make them feel how useful, agreeable, and ennobling, is the profession of agriculture, and, above all, how profitable the business must become when skilfully and economically carried on. These money-making considerations are, we suspect, the best moral guano that can be applied to the farmer's spiritual soil. The author writes well of the countryman's independence, the good effect of fresh salubrious air upon his health, and the moral influence of his every-day intimacy with nature upon his mind.

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